STEP

Mathematics

STEP **9465, 9470, 9475**

STEP Solutions

June 2007

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Step I, Solutions
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Section A: Pure Mathematics

1 There are many possible ways of answering this question, bnt it is essential to take a systematic and methodical approach. Here is one such strategy.

Consider the first two digits: there are eight possible sums (from $1 + 0 = 1$ to $4 + 4 = 8$).

If the first two digits sum to 1, there are two choices for the last two digits (1001 and 1010).

If the first two digits sum to 2 (11 and 20), there are three choices for the last two digits (because you can have 02 as well as 11 and 20).

If the first two digits sum to 3 (30, 21, 12), there are four choices for the last two digits.

If the first two digits sum to 4 (four choices), there are five choices for the last two digits.

If the first two digits sum to 5 (only four choices: 14, 23, 32, 41), there are the same four choices for the last two digits (there is no choice of 05 or 50).

If the first two digits sum to 6 (only three choices), there are three choices for the last two digits.

If the first two digits sum to 7 (only two choices), there are two choices for the last two digits.

If the first two digits sum to 8 (only one choice), there is one choice for the last two digits.

Hence there are $(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + (4 \times 4) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 70$ four-digit balanced numbers.

In general, there are $\sum_{r=1}^{k} r\,(r+1) + \sum_{r=k+1}^{2k} (2k+1-r)^2$ possible four-digit balanced numbers.

The second sum simplifies because $(2k+1-[k+1])^2 \equiv k^2$, $(2k+1-[k+2])^2 \equiv (k-1)^2$ and so on until $(2k+1-[2k])^2 \equiv 1$.)

Hence there are $\sum_{r=1}^{k} r\,(r+1) + \sum_{r=1}^{k} r^2$ possible four-digit balanced numbers.

$$\equiv 2\sum_{r=1}^{k} r^2 + \sum_{r=1}^{k} r$$

$$\equiv \frac{2k\,(k+1)\,(2k+1)}{6} + \frac{k\,(k+1)}{2} \text{ using the given identity for } \sum_{r=1}^{k} r^2$$

$$\equiv k\,(k+1)\left(\frac{4k+2+3}{6}\right) \equiv \frac{k\,(k+1)\,(4k+5)}{6}$$

2 **(i)** $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1$

$\Rightarrow A + B = \dfrac{\pi}{4}$, since A and B are both acute.

Now, if $\arctan\dfrac{1}{p} + \arctan\dfrac{1}{q} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{\frac{1}{p}+\frac{1}{q}}{1-\frac{1}{p}\times\frac{1}{q}} = 1$, by using the same identity for $\tan(A+B)$

$\Rightarrow \dfrac{q+p}{pq-1} = 1$, multiplying numerator and denominator by pq

$\Rightarrow pq - p - q - 1 = 0$

$\Rightarrow (p-1)(q-1) = 2$ (notice that this is analogous to "completing the square").

Since p and q are positive integers, we require two positive integers whose product is 2: the only choices are 1 and 2.

Hence $p = 2$ and $q = 3$ (or vice-versa: don't forget this second solution).

(ii) If $\arctan\dfrac{1}{r} + \arctan\dfrac{s}{s+t} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{1}{r} + \dfrac{s}{s+t} = 1 - \left(\dfrac{1}{r}\times\dfrac{s}{s+t}\right)$, by rearranging the expansion of $\tan(A+B) = 1$

$\Rightarrow s + t + rs = r(s+t) - s$, multiplying numerator and denominator by $r(s+t)$

$\Rightarrow s + t = rt - s$

$\Rightarrow 2s = t(r-1)$

$\Rightarrow r = \dfrac{2s}{t} + 1$

Since r is an integer, t must be a factor of $2s$. But the highest common factor of s and t is 1, so t cannot be a factor of s. Hence $t = 1$ or $t = 2$ (in the latter case, s is odd). Therefore either $r = 2s + 1$ or $r = s + 1$.

So, for all positive integer values of s,

$$\arctan\frac{1}{2s+1} + \arctan\frac{s}{s+1} = \frac{\pi}{4}$$

and for all positive odd integer values of s,

$$\arctan\frac{1}{s+1} + \arctan\frac{s}{s+2} = \frac{\pi}{4}$$

You might now like to consider whether the equation

$$\arctan\frac{1}{p} + \arctan\frac{1}{q} + \arctan\frac{1}{r} = \frac{\pi}{4}$$

has any integer solutions; first, you'll need an identity for $\tan(A+B+C)$.

3 There are many ways of tackling this question; here is one:

$$\cos^4\theta - \sin^4\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) \equiv \cos 2\theta$$

$$\cos^4\theta + \sin^4\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta - \sin^4\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos 2\theta \, d\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 0.$

Also $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta + \sin^4\theta \, d\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{1}{2}\sin^2 2\theta \, d\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{1}{4}\left(1 - \cos 4\theta\right) \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{4} + \frac{\cos 4\theta}{4} \, d\theta = \left[\frac{3\theta}{4} + \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{8}.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta = \frac{3\pi}{16}$

Adapting this idea:

$$\cos^6\theta - \sin^6\theta \equiv \left(\cos^2\theta - \sin^2\theta\right)^3 + 3\cos^2\theta\sin^2\theta\left(\cos^2\theta - \sin^2\theta\right) \equiv \cos^3 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta$$

$$\cos^6\theta + \sin^6\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)^3 - 3\cos^2\theta\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right) \equiv 1 - \frac{3}{4}\sin^2 2\theta$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta - \sin^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^3 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \sin^2 2\theta\right)\cos 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos 2\theta - \frac{1}{4}\sin^2 2\theta\cos 2\theta \, d\theta$$

$$= \left[\frac{\sin 2\theta}{2} - \frac{\sin^3 2\theta}{24}\right]_0^{\frac{\pi}{2}} = 0.$$

Also $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta + \sin^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{3}{4}\sin^2 2\theta \, d\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{3}{8}\left(1 - \cos 4\theta\right) \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{5}{8} + \frac{3\cos 4\theta}{8} \, d\theta = \left[\frac{5\theta}{8} + \frac{3\sin 4\theta}{32}\right]_0^{\frac{\pi}{2}} = \frac{5\pi}{16}.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta = \frac{5\pi}{32}$

You might like consider how you would prove that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2k}\theta - \sin^{2k}\theta \, d\theta = 0$ for all k, without having to derive a new identity for each value of k.

4 The beginning of this question is made much easier if you are able to factorise by inspection i.e. by looking at a product and filling in the missing terms by deduction rather than using a formal method such as long division.

For example, to factorise $x^3 + 2x + 12$ you should first see that $x = -2$ is a root of the expression (because $-8 - 4 + 12 = 0$) hence $x + 2$ is a factor (using the factor theorem), then think along the lines: $x^3 + 2x + 12 \equiv (x + 2) \times$ (a quadratic expression), and the coefficient of x^3 is 1 so the coefficient of x^2 must be 1; but that will create a term $2x^2$ which you don't want so there needs to be a $-2x$ term in the quadratic (which will generate a $-2x^2$ term when the brackets are expanded); but then there will be a $-4x$ term and you want $2x$ overall so you need a final term of 6 to create an additional $6x$; and $2 \times 6 = 12$ so the constant term will be correct. This takes time to write out, but is very quick to do in your head.

Factorising complex algebraic expressions by inspection is much easier than any other method: you should work carefully through the reasoning that tells you that

$$x^3 - 3xbc + b^3 + c^3 \equiv (x + b + c)\left(x^2 + b^2 + c^2 - xb - xc - bc\right).$$

Hence $2Q(x) \equiv 2x^2 + 2b^2 + 2c^2 - 2xb - 2xc - 2bc \equiv (x - b)^2 + (x - c)^2 + (b - c)^2$.

When we are told that k is a root of both equations, we can deduce that

$$ak^2 + bk + c = 0$$

and also that

$$bk^2 + ck + a = 0.$$

There are now a number of possible steps: an obvious one might be to use the quadratic formula to try to find an expression for k in terms of a, b and c. But notice that the expression we are asked to derive is still in terms of k, so any step which eliminates k is unlikely to be correct.

Instead, multiplying the first equation by b, the second by a and subtracting yields

$$\left(ac - b^2\right)k = bc - a^2.$$

Multiplying the first equation by c, the second by b and subtracting yields

$$\left(ac - b^2\right)k^2 = ab - c^2.$$

$$\Rightarrow \left(\frac{bc - a^2}{ac - b^2}\right)^2 = \frac{ab - c^2}{ac - b^2}$$

$$\Rightarrow \left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

$$\Rightarrow a^2bc - ab^3 - ac^3 + b^2c^2 = b^2c^2 + a^4 - 2a^2bc$$

$$\Rightarrow a^3 - 3abc + b^3 + c^3 = 0$$

$\Rightarrow (a + b + c)\left(a^2 + b^2 + c^2 - ab - ac - bc\right) = 0$, using the factorisation from the beginning of the question.

Hence either $a^2 + b^2 + c^2 - ab - ac - bc = 0$

$\Rightarrow (a-b)^2 + (a-c)^2 + (b-c)^2 = 0$

$\Rightarrow a = b = c$ (you should consider what can be deduced if a sum of squares is zero)

hence the equations are identical (and each reduces to $x^2 + x + 1 = 0$)

or $a + b + c = 0$

$\Rightarrow k = 1$.

You might like to consider what would happen if $ac = b^2$.

5 To tackle this question, a big, clear diagram is essential!

(i) Let the side length of the octahedron be $2k$.

Then the sloping "height" of a triangular face is $k\sqrt{3}$.

Also, the vertical height of the whole octahedron is $2k\sqrt{2}$.

Therefore, by the cosine rule, $8k^2 = 3k^2 + 3k^2 - 2 \times k\sqrt{3} \times k\sqrt{3} \times \cos A$.

Hence $A = \arccos\left(-\frac{2k^2}{6k^2}\right) = \arccos\left(-\frac{1}{3}\right)$.

(ii) The centre of each face is on any median of the equilateral triangle that is the face, and the centre is two-thirds of the way along the median from any vertex.

This is a quotable fact, but can be worked out from a diagram, using the fact that the centre of an equilateral triangle is equidistant from the three vertices: the centre divides the median in the ratio 1: $\cos 60°$.

The feet of the two medians from the apex of the octahedron in two adjacent triangles are $k\sqrt{2}$ apart.

Therefore, by similarity, adjacent centres of the triangular faces are $\frac{2}{3} \times k\sqrt{2}$ apart.

Therefore, the volume of the cube (whose vertices are the centres of the faces) is

$$\left(\frac{2}{3} \times k\sqrt{2}\right)^3 = \frac{16k^3\sqrt{2}}{27}$$

and the volume of the octahedron is

$$2 \times \frac{4k^2 \times k\sqrt{2}}{3} = \frac{8k^3\sqrt{2}}{3}$$

Hence the ratio of the volume of the octahedron to the volume of the cube is 9 : 2

6 **(i)** Since $x^2 - y^2 \equiv (x-y)(x+y)$, the given equation reduces to $d(x+y) = d^3$.

Hence $x + y = d^2$, and also we are given that $x - y = d$.

Therefore $x = \frac{1}{2}(d^2 + d)$ and $y = \frac{1}{2}(d^2 - d)$.

For the second equation, let $x = \sqrt{m}$ and $y = \sqrt{n}$ and let $d = 6$ (any choice of $d \geqslant 6$ will work).

Then $x = 21$ and $y = 15$, so $m = 441$ and $n = 225$.

You might like to check: $441 - 225 = 216 = (21 - 15)^3$.

(ii) It helps here to know that $x^3 - y^3 \equiv (x-y)(x^2 + xy + y^2)$.

Therefore the equation $x^3 - y^3 = (x-y)^4$ reduces to $x^2 + xy + y^2 = d^3$.

We know that $x^2 - 2xy + y^2 = d^2$ since $x - y = d$.

Hence, subtracting these two, $3xy = d^3 - d^2$.

This result can also be deduced by simplifying $x^3 - (x-d)^3 = d^4$ (using $x - y = d$).

Since $d = x - y$

$\Rightarrow 3x(x-d) = d^3 - d^2$

$\Rightarrow 3x^2 - 3dx - (d^3 - d^2) = 0$

$$\Rightarrow x = \frac{3d \pm \sqrt{9d^2 + 12(d^3 - d^2)}}{6} = \frac{3d \pm d\sqrt{12d - 3}}{6}$$

$$\Rightarrow 2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

For x to be integer we need $\dfrac{4d-1}{3}$ to be a perfect square.

If $d = 1$ then either $x = 0$ (not permitted) or $x = 1$ which implies $y = 0$ (not permitted).

So let $d = 7$ (for example), since $4 \times 7 - 1 = 27 = 3 \times 3^2$ ($d = 17$ also works).

$\Rightarrow 2x = 7 \pm 7\sqrt{9}$

Therefore $x = 14$ and hence $y = x - d = 7$ (choosing the positive values).

You might like to check: $14^3 - 7^3 = (2 \times 7)^3 - 7^3 = 7 \times 7^3 = 7^4 = (14 - 7)^4$.

7 When answering this question, it is useful to remember that the distance D between points with position vectors $\mathbf{p}$ and $\mathbf{q}$ is evaluated as $D = |\mathbf{p} - \mathbf{q}|$.

Hence if $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ then $D = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$

If you tackle a question such as this using a known formula (there is one for the minimum distance between two skew lines, using the vector product), you need to be careful with the subsequent algebra: if you cancel k from the numerator and denominator of a fraction, you are excluding the possibility that k might equal 0.

(i) $D^2 = (3 - 2\lambda + \mu)^2 + (-2 - 2\lambda + 2\mu)^2 + (7 + 3\lambda - 2\mu)^2$

$\equiv 17\lambda^2 + 9\mu^2 - 24\lambda\mu - 30\mu + 38\lambda + 62$

$\equiv (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36.$

The minimum value of D^2 is therefore 6, achieved by setting $\lambda = 1$ and $\mu = 3$.

When $\lambda = 1$ and $\mu = 3$ the points $(3\,, 2\,, -1)$ and $(7\,, 4\,, 3)$ are the minimal distance apart.

(ii) The strategy in part **(ii)** must be to adapt the strategy in part **(i)**:

$D^2 = (1 + 4k\beta)^2 + (-\alpha + \beta - \beta k)^2 + (-7 - 3\beta k)^2$

$\equiv 1 + 16\beta^2k^2 + 8k\beta + \alpha^2 + \beta^2 + \beta^2k^2 - 2\alpha\beta + 2\alpha\beta k - 2\beta^2 k + 49 + 9\beta^2k^2 + 42\beta k$

$\equiv 50 + 26\beta^2k^2 + 50\beta k + \alpha^2 + \beta^2 - 2\alpha\beta + 2\alpha\beta k - 2\beta^2 k$

$\equiv (\alpha - \beta + \beta k)^2 + (5\beta k + 5)^2 + 25.$

This is a useful simplification because the second bracket is only in terms of β and k.

Hence if $k \neq 0$ then the minimum distance between the two lines is 5 (when $\beta = -\frac{1}{k}$ and $\alpha = 1 - \frac{1}{k}$).

But if $k = 0$ then the minimum distance is $\sqrt{50}$ (when $\alpha = \beta$), and the lines are parallel: look at the two direction vectors.

You might like to consider what happens geometrically to the two lines as k tends to zero: notice that there is a "jump" from a minimum separation of 5 to a minimum separation of $\sqrt{50}$.

8 Consider $\mathrm{f}(x) = ax^3 - 6ax^2 + (12a+12)x - (8a+16)$.

Since $\mathrm{f}(2) = 8$ and $\mathrm{f}'(2) = 12$, the curve $y = \mathrm{f}(x)$ touches $y = x^3$ at $(2, 8)$: notice that both calculations are necessary to prove that the curves **touch**.

To find the other intersection point, let $\mathrm{f}(x) = x^3$

$\Rightarrow (a-1)x^3 - 6ax^2 + (12a+12)x - (8a+16) = 0$

Substituting $x = 2$: $8(a-1) - 24a + 2(12a+12) - (8a+16) \equiv 0$

$\Rightarrow (x-2)\left[(a-1)x^2 - (4a+2)x + (4a+8)\right] = 0$ (notice that factorising by inspection is much easier than using a method such as long division)

Substituting $x = 2$ into the quadratic factor: $4(a-1) - 2(4a+2) + (4a+8) \equiv 0$.

$\Rightarrow (x-2)(x-2)\left[(a-1)x - (2a+4)\right] = 0$

So the other intersection point has coordinates $\left(\dfrac{2a+4}{a-1}, \left[\dfrac{2a+4}{a-1}\right]^3\right)$

(i) When $a = 2$, $\dfrac{2a+4}{a-1} = 8$.

Hence the two graphs touch at $(2, 8)$ and intersect at $(8, 512)$.

$y = 2x^3 - 12x^2 + 36x - 32$ has no turning points: consider the derivative $6x^2 - 24x + 36 = 0$.

(ii) When $a = 1$, $\dfrac{2a+4}{a-1}$ is undefined.

Hence the two graphs touch at $(2, 8)$, and do not intersect elsewhere.

$y = x^3 - 6x^2 + 24x - 24$ has no turning points: consider the derivative $3x^2 - 12x + 24 = 0$.

(iii) When $a = -2$, $\dfrac{2a+4}{a-1} = 0$.

Hence the two graphs touch at $(2, 8)$ and intersect at $(0, 0)$.

$y = -2x^3 + 12x^2 - 12x$ turns when $x = 2 \pm \sqrt{2} \approx 3.4$ and 0.6: consider the derivative $-6x^2 + 24x - 12 = 0$.

It is essential that each sketch shows these features clearly, though the graphs need not be to scale.

Section B: Mechanics

9 A big, clear diagram is very helpful here - for you and also for the examiner!

Resolving parallel and perpendicular to the plane in both cases:

$X\cos\theta + \mu R = W\sin\theta \qquad R = X\sin\theta + W\cos\theta$

and

$kX\cos\theta = \mu R + W\sin\theta \qquad R = kX\sin\theta + W\cos\theta.$

$$\Rightarrow \frac{\cos\theta + \mu\sin\theta}{k\cos\theta - k\mu\sin\theta} = \frac{\sin\theta - \mu\cos\theta}{\mu\cos\theta + \sin\theta}$$

$$\Rightarrow \sin\theta\cos\theta + \mu\cos^2\theta + \mu\sin^2\theta + \mu^2\sin\theta\cos\theta = k\cos\theta\sin\theta - k\mu\sin^2\theta - k\mu\cos^2\theta + k\mu^2\sin\theta\cos\theta$$

$$\Rightarrow (k-1)\left(1+\mu^2\right)\sin\theta\cos\theta = \mu(k+1)$$

Since $\sin\theta\cos\theta \equiv \frac{1}{2}\sin 2\theta \leqslant \frac{1}{2}$

$$\Rightarrow \frac{\mu(k+1)}{(k-1)(1+\mu^2)} \leqslant \frac{1}{2}$$

$$\Rightarrow k\left(1+\mu^2\right) - 2\mu k \geqslant 1 + \mu^2 + 2\mu$$

$$\Rightarrow k\left(1 - 2\mu + \mu^2\right) \geqslant 1 + 2\mu + \mu^2$$

$$\Rightarrow k \geqslant \frac{(1+\mu)^2}{(1-\mu)^2}$$

Of course, the same result can be derived by initially resolving horizontally and vertically. In this case, the four equations are

$X + \mu R\cos\theta = R\sin\theta \qquad W = R\cos\theta + \mu R\sin\theta$

and

$kX = \mu R\cos\theta + R\sin\theta \qquad W = R\cos\theta - \mu R\sin\theta.$

From these, $\dfrac{X}{W} = \dfrac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} = \dfrac{1}{k}\left(\dfrac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}\right)$ which is equivalent to the above argument.

10 The displacement-time graph should have straight lines with equations:

$s = d - ut$ which is the path of the Norman army;

$s = xt$ which is the path of the Saxon horseman riding towards the Norman army;

$s = d - yt$ which is the path of the Norman horseman riding towards the Saxon army.

The Saxon horseman meets the Norman army when $xt = d - ut$,

i.e. at the point with coordinates $\left(\frac{d}{u+x}, \frac{dx}{u+x}\right)$.

Therefore his returu path has equation $s - \frac{dx}{u+x} = -x\left(t - \frac{d}{u+x}\right) \Rightarrow s = -xt + \frac{2dx}{u+x}$

The Norman horseman meets the Saxon arnuy when $s = 0$ i.e. when $t = \frac{d}{y}$.

Therefore his return path has equation $s = yt - d$

The subsequent analysis assumes that the two horsemen meet for the first time as they approach the opposing army, and meet for the second time as they retreat from the opposing army. If so, the horsemen first meet when $xt = d - yt \Rightarrow t = \frac{d}{x+y} \Rightarrow s = \frac{dx}{x+y}$ since $s = xt$.

They next meet when $-xt + \frac{2dx}{u+x} = yt - d$

$$\Rightarrow \frac{2dx}{u+x} + d = t(x+y) \Rightarrow t = \frac{d(u+3x)}{(x+y)(u+x)}$$

$$\Rightarrow s = -\frac{dx(u+3x)}{(x+y)(u+x)} + \frac{2dx}{u+x} = \frac{dx(2y-x-u)}{(u+x)(x+y)}.$$

Of course, this need not happen:

(i) If the Norman horseman rides quickly, he will meet the Saxon horseman for the second time $\left(\text{when } xt = yt - d \Rightarrow t = \frac{d}{y-x}\right)$ before the Saxon has reached the Norman army for the first time. If so, $\frac{d}{y-x} < \frac{d}{u+x} \Rightarrow u + x < y - x \Rightarrow u < y - 2x$

(ii) If the Saxon horseman rides quickly, he will meet the Norman horseman for the second time $\left(\text{when } -xt + \frac{2dx}{u+x} = d - yt \Rightarrow t = \frac{d(x-u)}{(u+x)(x-y)}\right)$ before the Norman has reached the Saxon army for the first time.

If so, $\frac{d(x-u)}{(u+x)(x-y)} < \frac{d}{y} \Rightarrow y(x-u) < (u+x)(x-y)$

$\Rightarrow 2xy < ux + x^2 \Rightarrow 2y - x < u$

11 Again, a big, clear diagram is very helpful here.

The acceleration up the slope is $-\frac{g}{2}$, hence the velocity at height $\frac{L}{2}$ is $\sqrt{u^2 - gL}$.

Then the particle travels freely as a projectile until it hits the floor, with a displacement of D horizontally and $-\frac{L}{2}$ vertically.

Hence $-\frac{L}{2} = \left(\sqrt{u^2 - gL}\right) t \sin 30^\circ - \frac{g}{2}t^2$ and $D = \left(\sqrt{u^2 - gL}\right) t \cos 30^\circ$

$\Rightarrow -L = t\sqrt{u^2 - gL} - gt^2$ and $2D = t\sqrt{3}\sqrt{u^2 - gL}$

$\Rightarrow -L = \frac{2D}{\sqrt{3}} - \frac{4gD^2}{3\left(u^2 - gL\right)}$

$\Rightarrow 4gD^2 - 2D\sqrt{3}\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$

$\Rightarrow 8gD\frac{\mathrm{d}D}{\mathrm{d}L} - 2\frac{\mathrm{d}D}{\mathrm{d}L}\sqrt{3}\left(u^2 - gL\right) + 2Dg\sqrt{3} - 3\left(u^2 - gL\right) + 3Lg = 0$

$\Rightarrow \frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\left(u^2 - gL\right)}.$

Since $R = D + \frac{L\sqrt{3}}{2} \Rightarrow \frac{\mathrm{d}R}{\mathrm{d}L} = \frac{\mathrm{d}D}{\mathrm{d}L} + \frac{\sqrt{3}}{2}$

Hence $\frac{\mathrm{d}R}{\mathrm{d}L} = 0 \Rightarrow \frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{\sqrt{3}}{2}.$

$\Rightarrow -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\left(u^2 - gL\right)} = -\frac{\sqrt{3}}{2}.$

$\Rightarrow -4gD\sqrt{3} + 3\left(u^2 - gL\right) + 2Dg\sqrt{3} - 3\left(u^2 - 2gL\right) = 0$

$\Rightarrow 3Lg = 2Dg\sqrt{3}$

$\Rightarrow 2D = L\sqrt{3}$

$\Rightarrow R = 2D = L\sqrt{3}.$

Substituting this into $4gD^2 - 2D\sqrt{3}\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$ we deduce that:

$3gL^2 - 3L\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$

$\Rightarrow gL = 2u^2 - 2gL$

$\Rightarrow 3gL = 2u^2$

$\Rightarrow R = \frac{2u^2}{g\sqrt{3}} \left(= L\sqrt{3}\right)$

Section C: Probability and Statistics

12 A tree diagram is very helpful when answering each part of this question.

(i) In the first case, $P(\text{the first sweet is red}) = \dfrac{a}{N}$

and $P(\text{the second sweet is red}) = \dfrac{a(a-1)}{N(N-1)} + \dfrac{(N-a)a}{N(N-1)} = \dfrac{aN-a}{N(N-1)} = \dfrac{a}{N}$

(ii) In the second case, $P(\text{the first sweet is red}) = \dfrac{pa}{N} + \dfrac{qb}{N}$

and $P(\text{the second sweet is red})$

$$= \frac{pa \times p(a-1)}{N(N-1)} + \frac{pa \times q(b+1)}{N(N+1)} + \frac{qb \times p(a+1)}{N(N+1)} + \frac{qb \times q(b-1)}{N(N-1)}$$

$$+ \frac{p(N-a) \times pa}{N(N-1)} + \frac{p(N-a) \times qb}{N(N+1)} + \frac{q(N-b) \times pa}{N(N+1)} + \frac{q(N-b) \times qb}{N(N-1)}$$

$$= \frac{pa}{N(N-1)}\Big[p(a-1) + p(N-a)\Big] + \frac{qb}{N(N-1)}\Big[q(b-1) + q(N-b)\Big]$$

$$+ \frac{pq}{N(N+1)}\Big[a(b+1) + b(a+1) + b(N-a) + a(N-b)\Big]$$

$$= \frac{pa}{N(N-1)}\Big[Np - p\Big] + \frac{qb}{N(N-1)}\Big[Nq - q\Big]$$

$$+ \frac{pq}{N(N+1)}\Big[Na + Nb + a + b\Big]$$

$$= \frac{p^2a}{N} + \frac{q^2b}{N} + \frac{pq(a+b)}{N}$$

$$= \frac{(p+q)(pa+qb)}{N}$$

$$= \frac{pa}{N} + \frac{qb}{N}$$

13 There are two ways of writing the answer to each part: one way is to think of a tree diagram, and then to combine the appropriate fractions; the other way is to count the number of "successful" combinations and divide by the number of "possible" combinations. Of course, these two approaches give the same numerical answer! You should make sure that you understand each version of the solution.

The probability that

(i) all four discs taken are numbered

$$= \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{{}^5C_4}{{}^{11}C_4} = \frac{1}{66}$$

(ii) all four discs taken are numbered given that the disc numbered "3" is taken first

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8}}{\frac{1}{11}} = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$$

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{\frac{1}{11}} \times 3 = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{1}{2}$$

because there are three equally likely outcomes once "3" has been taken first: "numbered, blank, another blank" or "blank, numbered, another blank" or "blank, another blank, numbered".

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{1 - \frac{10}{11} \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8}} \times 12 = \frac{{}^1C_1 \times {}^4C_1 \times {}^6C_2}{{}^{11}C_4 - {}^{10}C_4} = \frac{1}{2}$$

because there are twelve equally likely rearrangements of "3, another number, blank, another blank". Notice how P (disc numbered "3" taken) has been calculated on the denominator.

(v) exactly two numbered discs are taken, given that a numbered disc is taken first

$$= \frac{\frac{5}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{\frac{5}{11}} \times 3 = \frac{{}^5C_1 \times {}^4C_1 \times {}^6C_2}{{}^5C_1 \times {}^{10}C_3} = \frac{1}{2}$$

because there are three equally likely outcomes once a numbered disc has been taken first.

(vi) exactly two numbered discs are taken, given that a numbered disc is taken

$$= \frac{\frac{5}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{1 - \frac{6}{11} \times \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8}} \times 6 = \frac{{}^5C_2 \times {}^6C_2}{{}^{11}C_4 - {}^6C_4} = \frac{10}{21}$$

because there are six equally likely rearrangements of "number, another number, blank, another blank". Notice how P (a numbered disc is taken) has been calculated on the denominator.

14 $y = (x+1)\,\mathrm{e}^{-x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = -(x+1)\,\mathrm{e}^{-x} + \mathrm{e}^{-x} \equiv -x\mathrm{e}^{-x}$

Hence the graph turns (and crosses the y-axis) at $(0\,,1)$, and it crosses the x-axis at $(-1\,,0)$. As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y also tends to $-\infty$.

(i) $\mathrm{P}\,(X \geqslant 2) = 1 - p$

$\Rightarrow \mathrm{P}\,(X = 0) + \mathrm{P}\,(X = 1) = p$

$\Rightarrow (1+\lambda)\,\mathrm{e}^{-\lambda} = p.$

By considering the graphs $y = (x+1)\,\mathrm{e}^{-x}$ and $y = p$, we can see that this equation has a unique solution (for $0 < p < 1$).

(ii) $\mathrm{P}\,(X = 1) = q$

$\Rightarrow \lambda\mathrm{e}^{-\lambda} = q.$

The structure of part **(i)** suggests we consider the graphs $y = x\mathrm{e}^{-x}$ and $y = q$.

$y = x\mathrm{e}^{-x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x} - x\mathrm{e}^{-x} \equiv (1-x)\,\mathrm{e}^{-x}$

Hence the graph of $y = x\mathrm{e}^{-x}$ passes through $(0\,,0)$ and turns at $\left(1\,,\mathrm{e}^{-1}\right)$.

As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y also tends to $-\infty$.

Hence the equation $\lambda\mathrm{e}^{-\lambda} = q$ will have a unique solution when $\lambda = 1$ and $q = \mathrm{e}^{-1}$.

(iii) $\mathrm{P}\,(X = 1 \mid X \leqslant 2) = r$

$$\Rightarrow \frac{\lambda\mathrm{e}^{-\lambda}}{\mathrm{e}^{-\lambda} + \lambda\mathrm{e}^{-\lambda} + \frac{\lambda^2}{2!}\mathrm{e}^{-\lambda}} = r$$

$$\Rightarrow \frac{2\lambda}{2 + 2\lambda + \lambda^2} = r$$

The structure of parts **(i)** and **(ii)** suggests we consider the graphs $y = \dfrac{2x}{2 + 2x + x^2}$ and $y = r$.

The graph $y = \dfrac{2x}{2 + 2x + x^2}$ passes through $(0\,,0)$.

It turns when $2\left(2 + 2x + x^2\right) - 2x\,(2 + 2x) = 0 \Rightarrow 4 - 2x^2 = 0 \Rightarrow x = \pm\sqrt{2}$.

As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y tends to 0 from below.

Hence the equation $\Rightarrow \dfrac{2\lambda}{2 + 2\lambda + \lambda^2} = r$ will have a unique solution when $\lambda = \sqrt{2}$ and $r = \dfrac{2\sqrt{2}}{2 + 2\sqrt{2} + 2} = \dfrac{\sqrt{2}}{2 + \sqrt{2}} = \sqrt{2} - 1.$

Step II, Solutions
June 2007

Section A: Pure Mathematics

Q1 It is important to get off to a good start in any examination, especially so in STEPs, and Q1 is specifically designed to get as many candidates as possible off to such a start. Binomial series expansions are given in any of the permitted formulae books, and there is really no excuse for failing to pick up the marks on the introductory bit of the question. It is almost certainly to your advantage to simplify the terms of the expansion, but a little bit of care is in order here, else you are automatically losing accuracy marks later on in (a) and (b).

For part (a), you are told exactly what value of k to choose, and it is simply a case of using it **on both sides of the statement** – in the LHS to show that you can extract a sensible multiple of $\sqrt{3}$, and then in the RHS to see what you get as a decimal. Remember that working in powers of 10 makes the numerical working a lot simpler.

In (b), you have to choose a suitable value of k so that the LHS gives a multiple of $\sqrt{6}$. There is a small (but negative) integer value of k which will do this nicely. Many candidates, however, actually chose to work with $k = 50$ and, if you check, you will see that this *seems* to work equally well. However, the approximation gained is not nearly so accurate; this is because ….. ? Also, not a few candidates chose values of k greater than 100 in absolute value, and these are even worse, because ….. ?

In (ii), it is certainly possible to work back from the final answer in order to figure out what value of k to use here, but (again) you are looking for some (presumably) integer value that will this time yield a perfect cube multiple of 3 when $1 + \frac{k}{1000}$ is written as an improper fraction.

For interest's sake, the original version of the question used the first *three* terms of the series expansion with $k = 24$ to find an approximation to $\sqrt[3]{2}$.

Answers: (i) $1 + \frac{k}{200} - \frac{k^2}{80\,000} + \frac{k^3}{16\,000\,000}$; (a) 1.732 05; (b) 2.449 49.

Q2 It is fairly obvious that $x = p$ and $x = q$ are the two roots of the equation $\frac{dy}{dx} = 0$, which means that the derivative is a multiple of $(x - p)(x - q)$. Comparing the two then immediately gives b and c in terms of p and q. The sketch is a standard (positive) cubic, through the origin, with its two TPs in the first quadrant. Unintentionally, there are two possible candidates for the region $\boldsymbol{R}$, since the setters omitted to consider the one of them. Almost all candidates taking this paper identified the intended region, and this was because the question tries to get you to focus on the area around the point of inflection, which you are asked to mark on the diagram.

In (iii), m and n are simply the y-coordinates of the points corresponding to $x = p$ and q (respectively), and by this point you should know the curve's equation (in terms of p and q rather than b and c). Notice that $y(m)$ involves the extra qs and $y(n)$ involves extra ps, so the difference may just involve lots of $(q - p)$s, and the answer effectively tells you this much also. It may help in the working, both now and later, if you exploit this difference as much as possible.

Before embarking on the final part of the question, it would benefit you greatly to take a momentary pause and think about how the various bits of the question hang together. You were earlier asked to describe the symmetry of the cubic, and this was not just an idle bit of space-filler on the setter's part. Rather, it was an attempt to force you into recognising that the area of the region $\boldsymbol{R}$ can be found by means other than integration. Ignoring the coordinate axes on the diagram, and looking at the lines $x = p$, $x = q$, $y = n$ and $y = m$, you will see a nice rectangle appearing in the middle of the page. Because of the symmetry of the cubic, $\boldsymbol{R}$ is something to do with this rectangle, and this fact pretty much allows you to write the answer straight down, using the answer to (iii). On the other hand, if you want to do it by integration (as most candidates did) .....

And if you feel up to an algebraic challenge, see if you can work out, by integration, the area of the other possible region $\boldsymbol{R}$ – which also turns out – rather surprisingly, I felt – to be a rational multiple of $(q - p)^4$.

Answers: (i) $b = 3(p + q)$, $c = 6pq$; (ii) (two-fold) rotational symmetry about the P of I.

Q3 The first part of this question is a standard piece of bookwork, and requires only a modest ability to cope with substitution integration and a bit of trig. identity work. In (i) (a), you need to spot a suitable substitution for yourself – comparing the integrand with that in the introductory bit gives the game away, if you're stuck. In my day, the $t = \tan \frac{1}{2}x$ substitution was a very common bit of work, but you don't see it very often at A-level nowadays, so you could be forgiven for not being entirely familiar with it. Nonetheless, the principles of substitution still apply, and there may be the odd trig. identity to be employed, of course. The final two pieces of work here are greatly eased by the fact that they can be done in either direction. By that, I mean that one can eliminate all the *t*s in favour of *x*s, or vice versa. If you successfully complete part (i) (b), then (ii) is so much easier, since the only difference is that you must have $3 \sin^2 x$ in the denominator to give $14t^2$ instead of $6t^2$. Another simple substitution then changes the form into a standard arctan integral and, with a little bit of care, the whole thing can be wrapped up quite smoothly.

Overall, I would suggest that this is a fairly routine question, with no great leaps of thought required for a good A-level candidate to be able to work their way through it. What *is* required, however, is a high level of thoroughness and familiarity with the basic techniques of the trig. and calculus involved therein. Such capabilities are an essential requirement if you are preparing for future STEPs.

Answers: (i) (a) $\frac{\pi}{4}$; (ii) $\frac{\pi}{6\sqrt{3}}$.

Q4 This was actually not a particularly popular, or well done, question, although I still maintain that it is quite an easy one when it comes down to it! To begin with, it is really, *really* obvious that you need to expand the given trig. expressions using the *Addition Formulae*. Then, in order to obtain tans throughout, rather than sines and cosines, you are going to have to divide by ..... (hint: note the introductory conditions at the very start of the question, which are given to enable you not to worry about dividing by .....). Wangling it into the given form and checking that the given condition holds is not much more than an algebraic exercise at this stage, and shouldn't prove too much of a burden. However, it is easy to overlook the fact that you are asked to prove an "if and only if" statement, which is two-directional. In point of fact, it is the case here that a clear line of reasoning from first equation to final one actually *is* entirely reversible, although it is best to (at least) point out that this is so, rather than ignore it.

For the next three parts, see how this result can now be used to solve each of the given equations, once the "*A*" and the "*B*" have been clearly identified. Also, don't forget to identify the α, β and γ (the same in each of the three parts) and verify that $\alpha^2 = \beta^2 + \gamma^2$. It is, of course, perfectly possible to start each bit from scratch, and the wording of the question doesn't actually prevent you from doing so, but it would seem a bit of a waste of time and effort to do so. Having said that, several candidates successfully did (iii) by collecting the two ($3x$) terms up together and collecting them up in an $R\sin(3x+\theta)$ form.

Incidentally, my favourite part of the question was (ii), in which I got to play a bit of a dirty trick – the statement looks like an equation, but is actually an ..... because .....

Answers: (i) $\frac{2\pi}{3}$, $\frac{5\pi}{3}$; (ii) all $x \in [0, 2\pi)$; (iii) $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$ and $\frac{\pi}{3}$, $\frac{4\pi}{3}$.

Q5 Part (i) is a standard opener using compositions of functions, and the algebra shouldn't prove too demanding if you're careful. Again, simplified answers at each stage are most helpful for successful further progress through a question like this. The sequence of powers of f turns out to be periodic with period 3, and so f^{2007} isn't quite the big ask that it might seem to be at first sight.

As you're told what to do in (ii), it is just a case of being careful in establishing the relationship. A grasp of the process of mathematical induction is an essential requirement for STEP II, even if it is no longer on single Maths syllabuses elsewhere, and this could be used in this case. An informal inductive proof was perfectly acceptable also, although it was equally acceptable to establish the cases for $n = 1$, 2 and 3 and then point out that the periodicity of the tan function guarantees the rest.

Now, part (iii) offers something a little more demanding. The simple approach involves spotting that the use of $t = \sin\theta$ gives $\sqrt{1-t^2} = \cos\theta$, and then a similar inductive argument to (ii)'s will lead to an admittedly unappealing but otherwise simple result for g^n in a $\sin(A+B)$ kind of way. However, if instead you note that $\sqrt{1-t^2}$ denotes the **positive** square-root of $1-t^2$, which may actually be $-\cos\theta$ for some values of θ (and hence t). Thus, in fact, g^2 *can* turn out to be just x again, so that the sequence $\{g, g^2, g^3, \ldots\}$ turns out to be oscillating (i.e. periodic with period 2). If you proceed further down this route, exploring which parts of g's domain give what "powers" of g, you get very interesting results which may be worth discussion, but were not expected under examination conditions here.

Answers: (i) $f^2(x) = \frac{x-\sqrt{3}}{1+\sqrt{3}\,x}$, $f^3(x) = x$; $f^{2007}(x) = x$.

(iii) Answer 1: $g^n(t) = \sin(\sin^{-1} t + \frac{n\pi}{6})$; Answer 2: $g^n(t) = \begin{cases} g(t) & n \text{ odd} \\ t & n \text{ even} \end{cases}$.

Q6 Once again, this starts off with a bit of very basic work that a realistic STEP candidate needs to be in a position to rattle off quickly and efficiently. The "Hence" at the start of line 2 of the question tells you that the answer to this integral is to be found in the two previous answers, without further calculus work being done. It is, therefore, very bad examination practice to ignore the "Hence" demand and go off on an "or otherwise" route that isn't actually needed. And there's a strong chance you may not get any marks at all for your alternative approach.

In (ii), there is no reason why you can't treat the given differential equation as a quadratic in $\frac{dy}{dx}$ and solve it to get two slightly different, and much simpler, differential equations than the original one. At this stage, if you have your wits about you, and you are **NOT** getting a $\sqrt{3+x^2}$ anywhere in sight, then you really ought to be a bit suspicious about why not! For the rest of it, it is a simple case of integrating using (i)'s result, and then applying the given initial condition to find the constants of integration in each of the two cases.

Answers: (i) $\frac{1}{\sqrt{3+x^2}}$, $\frac{2x^2+3}{\sqrt{3+x^2}}$; $\frac{1}{2}x\sqrt{3+x^2} + \frac{3}{2}\ln\left(x+\sqrt{3+x^2}\right)$ $(+\,C)$.

(ii) $y = \frac{1}{6}x\sqrt{3+x^2} + \frac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \frac{1}{6}x^2 - \frac{1}{6} - \frac{1}{2}\ln 3$;

$y = -\frac{1}{6}x\sqrt{3+x^2} - \frac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \frac{1}{6}x^2 + \frac{1}{2} + \frac{1}{2}\ln 3$.

Q7 I like this question, although I accept that lots of candidates were probably put off by a question that looks like something they've never seen before. However, it is often the case that questions of the "new and weird-looking kind" can actually turn out to be relatively easy IF you're prepared to be a bit adventurous.

The opening bit introduces you to a (possibly) new idea, and then gets you to practise this idea in a couple of cases in order that you get the hang of it. Then, in part (i), you actually get to use one of these ideas, and you're pretty much told exactly what to do, and which of the two initial functions to use to get the given result.

Next, in (ii), you're thrown in the deep-end rather more and left to decide what to do for yourself. Here, however, there is reference made to a mysterious "suitable function" to be used. Now, if you believe that the setter is out to trap you, trick you, and grind you into the ground then you probably think you're all on your own at this stage and have to find your own function. But you're wrong! The setters are actually trying to give you every opportunity to do some good mathematics, and every effort is made to point you in the right direction if is felt at all suitable to do so. In this case, you were initially asked to show that the sin and ln functions had the property being referred to. Then you used the sin function in (i). Perhaps, just *perhaps*, you are meant to be using the other one in (ii). If you can use the ln function to establish this next result (called the *Arithmetic Mean – Geometric Mean Inequality*), then parts (a) and (b) at the end simply use it twice; once with very little thought required, and one with a little more thought needed. Be brave! Give it a go.

Answers: (ii) -2.

Q8 If you don't know what is wanted in (i), then you really shouldn't be doing this question. It also really helps if you realise that if s and t are positive, then X is the point between B and C such that $BX : XC = t : s$. Once you have these ideas in place, this question involves nothing more than finding the points of intersection referred to, by equating two different line equations at a time. You will need to introduce a new pair of parameters each time, but if you keep each stage of working separate, then there is no reason not to use the same two symbols each time; and then solve pairs of simultaneous equations, gained by equating the **b**– and **c**–components of the two relevant line vector equations, for these two parameters in terms of β and γ. The result displayed is known as *Ceva's Theorem*.

Answers: (i) The straight line through B and C.

Section B: Mechanics

Q9 The greatest problem with marking mechanics questions on the STEPs is that candidates seem to be so unwilling, or unable, to mark up a decently labelled diagram with all relevant forces on them, or, in this case, relevant velocities and angles. On the face of it, this is just a collisions question dressed up a bit, and there really are only the two mechanical principles to be applied here: *Conservation of Linear Momentum* (CLM) and *Newton's Experimental Law of Restitution* (NEL or NLR). If you take a side-on view of the cone, then the collision – at the moment of impact – is effectively the same as would be given by a plan view of a particle striking a vertical wall: directly, in the first instance, and then obliquely in the second. Applying CLM parallel to this line of impact (which is very easy in the first case and, in fact, the reason why you were asked for an explanation to begin with so that you were pointed in the right direction) and NEL perpendicular to it are essential steps in both parts of the question. In order to prevent you worrying about how the cone might bounce off the plane, you are told that this does not happen. So there is no point considering CLM vertically for the particle-cone collision, but there is still the horizontal motion of the cone to consider.

In (ii), the collision is oblique to the line of the cone's side, so there are two angles involved, and a bit of trig. work might be needed to sort things out. Alternatively, rather than re-doing (i)'s working in this separate case, one could simply consider the components of the "incoming" velocity, and the second answer for w is exactly the same as the first, but with u replaced by ..... For the very final part of the question, a little calculus is in order.

Q10 The first thing to do here is to find the position of the centre of mass of the composite figure, and this is fairly easily done by taking moments about some suitable point. Most candidates who actually attempted this question then went very badly astray, largely due to lack of a discernible approach in their jottings. In slipping-tilting situations, the standard approach is to examine separately what happens at the instant when slipping occurs *assuming* that tilting hasn't, and then to examine what happens when tilting occurs *assuming* that the slipping hasn't. This then gives two sets of conditions on P which can be compared. Remember that P can be in either direction, hence the modulus sign in the answer, which needs to be explained somewhere along the line.

Q11 In a similar sort of way to Q9, this is just a reasonably standard projectiles question dressed up a bit, and the vector set-up should help you work in the third-dimension quite naturally. The given answer in (i) should help confirm that you're doing the right thing to begin with (or not!). Completing the square, or differentiating, will give the value of t when OP is a minimum, and this should then turn out to be the same instant/position as can be found in part (ii) by differentiating the vertical ($\mathbf{k}$ –) component of the displacement vector.

Part (iii) can be done in a couple of ways: one is very lengthy, pressing on with the vector formulation for as long as possible, but the intended approach is to work with distance and time as scalars on the assumption that the bullet moves in a straight line.

Answers: (i) $\underline{\mathbf{r}} = \left(50 - 5t\sqrt{5}\right)\mathbf{i} + \left(5t\sqrt{15}\right)\mathbf{j} + \left(5t\sqrt{5} - 5t^2\right)\mathbf{k}$;

$\underline{\mathbf{p}} = \dfrac{75}{2}\mathbf{i} + \dfrac{25\sqrt{3}}{2}\mathbf{j} + \dfrac{25}{4}\mathbf{k}$; (0)60°.

Section C: Probability and Statistics

Q12 With given answers, as here, it is important to make your method clear, since there is a lot of fiddling going on as candidates inevitably manage to wangle this answer somehow. Splitting the required event into a series of mutually exclusive events and recognising which of these events are independent, is crucial, and it helps both you and the examiner if there is an accompanying (brief!) explanation as to what you are doing. It seems to me that the first part can be approached in at least a couple of obvious ways. Firstly, one could work out the prob. that one die gives at least one 6 in the first r throws, $P(r)$, say, and then observing that the prob. that both dice have given a 6 **at** the r^{th} throw is $P(r) - P(r - 1)$. Alternatively, one could write it as the sum of the probs. that {neither dice has recorded a 6 in the first $r - 1$ throws and then both give 6s} with the prob. that {one die gives a 6 before the r^{th} throw and then the 2^{nd} die first gives a 6 on the r^{th} throw}.

Finding the expected value of the number of throws is routine, in principle at least, and you are given a result to use to help you with this, if needed. In (ii), equating this expression (in terms of p only) to m and then re-arranging gives a ..... equation in p, which should now be very familiar territory.

Answers: (i) $p = \frac{1}{m}\left\{m + 1 - \sqrt{m^2 - m + 1}\right\}$.

Q13 The first couple of terms of the series expansion for e gives the opening result, which you are obviously intended to exploit later on in the question. Next, p(at least 1 matching pair) is best considered in the form 1 – p(no matching pairs), and you get (with a little imagination) a whole load of fractions of the form $\frac{n-r}{n}$ which can be approximated by the exponential result given at the outset. The laws of indices and a bit of summation of an AP then sort out the rest of the first problem.

The next two parts each involve working with an inequality, and the second requires another use of the initial exponential result. Each employs the remarkably accurate rational approximation to ln 2 given in the question.

Answers: 23 ; 253.

Q14 The pdf sketch in (i) consists of three (actually, five – don't forget to indicate clearly the zero bits!) pieces. Then, equating expressions for the endpoints of these pieces, which are defined in two different ways, immediately gives a and b in terms of k. After this, equating the total area under this graph to 1 (total probability) then gives the exact value of k, and hence a and b also. This is the bulk of the question done, and most of it is really pure mathematical content.

The last part is similar in content, requiring – in statistical terms – only the observation that m is given by $\int_1^m f(x) = \frac{1}{2}$. Now, it is not immediately clear which piece of the function that m lies in, so a little bit of justification needs to be given to explain the relevance of any subsequent working that you give. Some fairly simple approximations for e should enable you to show that m is **not** in the first piece but **is** in the second.

Answers: (ii) $a = 2\ln k$, $b = \frac{\ln k}{2k}$; $k = e^{1/3}$, $a = \frac{2}{3}$, $b = \frac{1}{6}e^{-1/3}$; (iii) $m = 3(e^{1/3} - ½)$.

Step III, Solutions
June 2007

Section A: Pure Mathematics

1. The first result can be obtained by applying a compound angle formula to $\tan\left((\theta_1+\theta_2)+(\theta_3+\theta_4)\right)$ and then repeating the application to each of $\tan(\theta_1+\theta_2)$ and $\tan(\theta_3+\theta_4)$ where they appear. On simplification, this gives

$$\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{t_1+t_2+t_3+t_4-t_2t_3t_4-t_3t_4t_1-t_4t_1t_2-t_1t_2t_3}{1-t_1t_2-t_1t_3-t_1t_4-t_2t_3-t_2t_4-t_3t_4+t_1t_2t_3t_4}.$$

As t_1, etc are the roots of the equation $at^4+bt^3+ct^2+dt+e=0$, then $at^4+bt^3+ct^2+dt+e=a(t-t_1)(t-t_2)(t-t_3)(t-t_4)$, which yields, from expansion and comparison of coefficients, the four results

$$t_1+t_2+t_3+t_4=\frac{-b}{a},\quad t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4=\frac{c}{a},$$

$$t_2t_3t_4+t_3t_4t_1+t_4t_1t_2+t_1t_2t_3=\frac{-d}{a},\text{ and }\ t_1t_2t_3t_4=\frac{e}{a}.$$

These substituted in the first result lead to $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\dfrac{-b+d}{a-c+e}$.

Applying double and compound angle formulae to $p\cos 2\vartheta+\cos(\vartheta-\alpha)+p=0$ gives the equation $2p\cos^2\vartheta+\cos\vartheta\cos\alpha+\sin\vartheta\sin\alpha=0$, which can be rearranged as

$$\cos\alpha+\tan\vartheta\sin\alpha=\frac{-2p}{\sec\vartheta}.$$

Squaring this and replacing $\tan\vartheta$ by t, $(\cos\alpha+t\sin\alpha)^2=\dfrac{4p^2}{1+t^2}$.

Rearranging this obtains the quartic equation $t^4\sin^2\alpha+t^3\sin 2\alpha+t^2+t\sin 2\alpha+(\cos^2\alpha-4p^2)=0$, and so, from the second result $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\dfrac{0}{-4p^2}=0$, and thus $\theta_1+\theta_2+\theta_3+\theta_4=n\pi$.

2. (i)

$$1.3.5.7.....(2n-1)=\frac{1.2.3.4....2n}{2.4.6.8....2n}==\frac{(2n)!}{2.1.2.2.2.3.2.4.....2.n}=\frac{(2n)!}{2^n.1.2.3.4.....n}=\frac{(2n)!}{2^n n!}$$

Using the binomial theorem, which is valid given the condition $|x|<\frac{1}{4}$,

$$(1-4x)^{\frac{-1}{2}}=1+\frac{-1}{2}(-4x)+\frac{\frac{-1}{2}\frac{-3}{2}(-4x)^2}{2!}+...$$

$$=1+1.(2x)+\frac{1.3}{2!}(2x)^2+...+\frac{1.3.5.7...(2n-1)}{n!}(2x)^n+...$$

So the first result of the question yields $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{\frac{(2n)!}{2^n n!}(2x)^n}{n!}$ leading to the required expression.

(ii) Differentiating $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!x^n}{(n!)^2}$ with respect to x, and multiplying the result by x gives $\frac{2x}{(1-4x)^{\frac{3}{2}}}=\sum_{n=1}^{\infty}\frac{(2n)!x^n}{n!(n-1)!}$ and substituting $x=\frac{6}{25}<\frac{1}{4}$, gives the desired result.

(iii) Integrating $(1-4x)^{\frac{-1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n)!x^n}{(n!)^2}$ with respect to x, gives $\frac{-1}{2}(1-4x)^{\frac{1}{2}}=x+\sum_{n=1}^{\infty}\frac{(2n)!x^{n+1}}{(n+1)!n!}+c$, and substituting $x=0<\frac{1}{4}$, gives $c=\frac{-1}{2}$.

Now substituting $x=\frac{2}{9}=\frac{2}{3^2}<\frac{1}{4}$ and simplifying, gives the desired result.

3. (i) $F_3=2, F_4=3, F_6=5, F_6=8, F_7=13, F_8=21$

(ii) The result requires no term beyond F_{2k+2} should appear on the RHS so the first strategy is to replace F_{2k+3} and hence

$$F_{2k+3}F_{2k+1}-F_{2k+2}{}^2=(F_{2k+2}+F_{2k+1})F_{2k+1}-F_{2k+2}{}^2=(F_{2k+1}-F_{2k+2})F_{2k+2}+F_{2k+1}{}^2=-F_{2k}F_{2k+2}+F_{2k+1}{}^2$$

as required.

(iii) The initial case is trivial to demonstrate, and so the induction runs from assuming that $F_{2k+1}F_{2k-1}-F_{2k}{}^2=1$, and attempting to prove that

$F_{2(k+1)+1}F_{2(k+1)-1}-F_{2(k+1)}{}^2=1$.

$F_{2(k+1)+1}F_{2(k+1)-1}-F_{2(k+1)}{}^2=F_{2k+3}F_{2k+1}-F_{2k+2}{}^2=-F_{2k}F_{2k+2}+F_{2k+1}{}^2$ from (ii)

$=-(-F_{2k-1}F_{2k+1}+F_{2k}{}^2)$ by a similar argument to (ii) $=-(-1)$ by inductive hypothesis.

The deduction follows from adding $F_{2n}{}^2$ to both sides of the result just proved.

(iv) This result cannot be deduced directly from (iii) as the nature of the expression differs in the type of subscript. Thus consider

$$F_{2n-1}{}^2+1=\left(F_{2n+1}-F_{2n}\right)^2+1=F_{2n+1}{}^2-2F_{2n+1}F_{2n}+F_{2n}{}^2+1=F_{2n+1}{}^2-2F_{2n+1}F_{2n}+F_{2n-1}F_{2n+1}$$

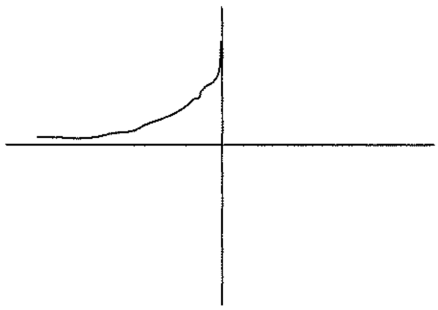
from (iii) and hence the desired result is obtained.

4.

$$y=a\sin t\Rightarrow\dot{y}=a\cos t$$

$$x=a\left(\cos t+\ln\tan\frac{t}{2}\right)\Rightarrow\dot{x}=a\left(-\sin t+\frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right)=a(-\sin t+\operatorname{cosec}t)=a\cos t\cot t$$

giving $\frac{dy}{dx}=\tan t$.



(y intercept a, y axis tangential to curve, x axis asymptote)

Tangent is $y-a\sin t=\tan t\left(x-a\left(\cos t+\ln\tan\frac{t}{2}\right)\right)$ giving Q as $\left(a\ln\tan\frac{t}{2},0\right)$ and thus $PQ=\sqrt{\left((a\cos t)^2+(a\sin t)^2\right)}=a$

$$\dot{y}=a\cos t\Rightarrow\ddot{y}=-a\sin t$$

$$\dot{x}=a(-\sin t+\operatorname{cosec}t)\Rightarrow\ddot{x}=a(-\cos t-\operatorname{cosec}t\cot t)$$

$$\dot{x}^2+\dot{y}^2=(a\cos t\cot t)^2+(a\cos t)^2=a^2\cot^2 t$$

$$\dot{x}\ddot{y}-\dot{y}\ddot{x}=a\cos t\cot t\times -a\sin t-a\cos t\times a(-\cos t-\operatorname{cosec}t\cot t)$$

$$=a^2\left(-\cos^2 t+\cos^2 t+\cot^2 t\right)=a^2\cot^2 t$$

giving $\rho=a\cot t$.

From the results for $\frac{dy}{dx}$ and ρ, C is

$$\left(a\left(\cos t + \ln\tan\frac{t}{2}\right) - \rho\sin t, a\sin t + \rho\cos t\right) = \left(a\ln\tan\frac{t}{2}, a\,\mathrm{cosec}\,t\right)$$

Which has the same x coordinate as Q.

5. $\frac{dr}{dx} = x\left(x^2 - 1\right)^{\frac{-1}{2}} = \cosh\theta$

$y = \ln r^2 = 2\ln r$

So $\frac{dy}{dx} = \frac{2}{r}\frac{dr}{dx} = \frac{2\cosh\theta}{r}$

$\frac{dx}{d\theta} = -\mathrm{cosech}^2\theta$ and $r = \mathrm{cosech}\theta$,

So differentiating the previous result and substituting,

$$\frac{d^2y}{dx^2} = \frac{2r\sinh\theta\frac{d\theta}{dx} - 2\cosh\theta\frac{dr}{dx}}{r^2} = \frac{2\left(\mathrm{cosech}\theta\sinh\theta \times -\sinh^2\theta - \cosh\theta\cosh\theta\right)}{r^2} = -\frac{2\cosh 2\theta}{r^2}$$

Similarly,

$$\frac{d^3y}{dx^3} = -\frac{2r^2 2\sinh 2\theta\frac{d\theta}{dx} - 2\cosh 2\theta \times 2r\frac{dr}{dx}}{r^4} = \frac{4}{r^4}(\sinh 2\theta + \cosh 2\theta\coth\theta) = \frac{4}{r^3}\cosh 3\theta$$

In order to hypothesise a result for $\frac{d^n y}{dx^n}$, the important thing is to appreciate that the 4 has come from 2 times exponent of r and multiple of θ.

So $\frac{d^n y}{dx^n} = 2 \times (-1)^{n-1}\frac{(n-1)!}{r^n}\cosh n\theta$ which may be proved by induction, the inductive differentiation step following the same pattern of working as used for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

6. $pp^* = qq^* = a^2$

and so $a^2(p-q) = qq^*p - pp^*q = -pq\left(p^* - q^*\right)$ and hence the required result.

If PQ and RS are perpendicular then $p - q = ki(r - s)$ for some real k, and thus

$p^* - q^* = -ki\left(r^* - s^*\right)$, and so $pq = -a^2\frac{p-q}{p^*-q^*} = a^2\frac{r-s}{r^*-s^*} = -rs$

For $n = 3$, $B_1B_2 \perp A_1A_2$ etc. $\Rightarrow a_1a_2 + b_1b_2 = 0$ etc.

Thus $b_1{}^2 = \frac{b_1b_2 \times b_1b_3}{b_2b_3} = \frac{-a_1a_2 \times -a_1a_3}{-a_2a_3} = -a_1{}^2$ and so $b_1 = \pm ia_1$

i.e. two choices of B_1.

For $n=4$, $B_1B_2 \perp A_1A_2$ etc. $\Rightarrow a_1a_2+b_1b_2=0$ etc. but this only yields 3 independent equations as e.g. $a_3a_4+b_3b_4=0$ can be obtained from the other three equations by $a_3a_4=\frac{a_2a_3\times a_4a_1}{a_2a_1}$ etc. Hence there are arbitrarily many possible choices for B_1.

For $n>4$, the corresponding results are as for $n=3$ or $n=4$ depending on whether n is odd or even.

7. (i) $u=v^{-1}\Rightarrow \frac{du}{dv}=-v^{-2}$ so $t(x)=\int_{\infty}^{\frac{1}{x}}\frac{1}{1+v^{-2}}\times -v^{-2}dv=\int_{\frac{1}{x}}^{\infty}\frac{1}{v^2+1}dv$

so $t\left(\frac{1}{x}\right)+t(x)=\int_0^{\frac{1}{x}}\frac{1}{1+u^2}du+\int_{\frac{1}{x}}^{\infty}\frac{1}{v^2+1}dv=\int_0^{\infty}\frac{1}{1+u^2}du=\frac{1}{2}p$

Letting $x=1$ gives the desired result.

(ii) $y=\frac{u}{\sqrt{1+u^2}}\Rightarrow u=\frac{y}{\sqrt{1-y^2}}$

so $\frac{du}{dy}=\frac{(1-y^2)^{\frac{1}{2}}-y\times -y(1-y^2)^{\frac{-1}{2}}}{1-y^2}=\frac{(1-y^2)+y^2}{(1-y^2)^{\frac{3}{2}}}$ and hence the result.

Using the given substitution for u,

$$t(x)=\int_0^{\frac{x}{\sqrt{1+x^2}}}\frac{1}{1+\frac{y^2}{1-y^2}}\times\frac{1}{(1-y^2)^{\frac{3}{2}}}dy=\int_0^{\frac{x}{\sqrt{1+x^2}}}\frac{1}{(1-y^2)^{\frac{1}{2}}}dy=s\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Again letting $x=1$, and using the result from part (i) gives the desired result.

(iii) $z=\frac{u+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}u}\Rightarrow u=\frac{z-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}z}\Rightarrow\frac{du}{dz}=\frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2}$

Using this substitution,

$$t(x)=\int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}}\frac{1}{1+\left(\frac{z-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}z}\right)^2}\times\frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2}dz=\int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}}\frac{\frac{4}{3}}{\left(1+\frac{1}{\sqrt{3}}z\right)^2+\left(z-\frac{1}{\sqrt{3}}\right)^2}dz=\int_{\frac{1}{\sqrt{3}}}^{\frac{x+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}x}}\frac{1}{1+z^2}dz$$

Letting $x=\frac{1}{\sqrt{3}}$ gives the required result.

By definition $t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+u^2}du$, by the previous result just obtained

$t\left(\frac{1}{\sqrt{3}}\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}dz$, and from part (i) $t\left(\frac{1}{\sqrt{3}}\right) = \int_{\sqrt{3}}^{\infty} \frac{1}{1+v^2}dv$ and so adding these three

results gives $3t\left(\frac{1}{\sqrt{3}}\right) = \int_0^{\infty} \frac{1}{1+u^2}du = \frac{1}{2}p$

8. (i) Substituting each u into the differential equation yields simultaneous equations $a(x) + xb(x) = 0$ and $e^{-x}(1 - a(x) + b(x)) = 0$ which solve to give

$a(x) = \frac{x}{1+x}$ and $b(x) = \frac{-1}{1+x}$

The general solution is $u = Ax + Be^{-x}$.

$y = \frac{1}{3u}\frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{3u^2}\left(\frac{du}{dx}\right)^2 + \frac{1}{3u}\frac{d^2u}{dx^2}$ which when substituted into equation (*), multiplied by 3u, and collected on one side gives the required result.

$u = Ax + Be^{-x} \Rightarrow \frac{du}{dx} = A - Be^{-x} \Rightarrow y = \frac{A - Be^{-x}}{3(Ax + Be^{-x})}$,

and substitution of $x = 0, y = 0$ gives $A = B$ and hence $y = \frac{1 - e^{-x}}{3(x + e^{-x})}$.

(ii) Substituting $y = \frac{1}{u}\frac{du}{dx}$ into the given equation yields

$\frac{d^2u}{dx^2} + \frac{x}{1-x}\frac{du}{dx} - \frac{1}{1-x}u = 0$ which is the equation in the first part with x replaced by $-x$

So the general solution is $u = Cx + De^x$

Substitution of $x = 0, y = 2$ again gives $A = B$, and hence $y = \frac{1 + e^x}{x + e^x}$

Section B: Mechanics

9. Conservation of energy leads to the equation

$2\left[\frac{1}{2}m\left(a\dot{\theta}\right)^2\right] + mk^2a^2(\theta - \alpha)^2 = mk^2a^2(\beta - \alpha)^2$ which, when simplified, and working in the variable $(\theta - \alpha)$ rather than θ can be rearranged as

$(\dot{\theta} - \alpha) = k\sqrt{\left((\beta - \alpha)^2 - (\theta - \alpha)^2\right)}$.

Separating the variables and performing the standard integral yields $\theta - \alpha = (\beta - \alpha)\sin(kt + \phi)$ (it does not matter that $(\beta - \alpha) < 0$).

The initial position from which the system is released gives $\phi = \frac{\pi}{2}$ and so $\theta = \alpha + (\beta - \alpha)\cos kt$.

The three possibilities that can arise are that $\dot{\theta} = 0, \theta < \frac{\pi}{2}$, that $\dot{\theta} = 0, \theta = \frac{\pi}{2}$, or that $\dot{\theta} > 0, \theta = \frac{\pi}{2}$.

The first of these is SHM and has period $\frac{2\pi}{k}$, which occurs if $\alpha - (\beta - \alpha) < \frac{\pi}{2}$ i.e. if $\beta > 2\alpha - \frac{\pi}{2}$.

For the second case, oscillations do not occur. Then,

$\dot{\theta} = 0 \Rightarrow \sin kt = 0 \Rightarrow \cos kt = -1$ (not $\cos kt = 1$ as this is the initial position) and so $\frac{\pi}{2} = \alpha - (\beta - \alpha)$ i.e. $\beta = 2\alpha - \frac{\pi}{2}$.

The third case is partially SHM until $\theta = \frac{\pi}{2}$ and then the motion is reflected.

So a quarter of the period is given by $\frac{\pi}{2} = \alpha + (\beta - \alpha)\cos kt$ and hence the period is

$\frac{4}{k}\cos^{-1}\left(\frac{\frac{\pi}{2} - \alpha}{\beta - \alpha}\right)$ which occurs if $\beta < 2\alpha - \frac{\pi}{2}$.

10. Using uniform acceleration formulae with $(\ddot{x}, \ddot{y}) = (-g\sin\phi, -g\cos\phi)$, then

$(x, y) = \left(Vt\cos\theta - \frac{1}{2}gt^2\sin\phi, Vt\sin\theta - \frac{1}{2}gt^2\cos\phi\right)$.

To return on the same path $\dot{x} = 0$ when $y = 0$. So $t = \frac{V\cos\theta}{g\sin\phi} = \frac{2V\sin\theta}{g\cos\phi}$

i.e. $2\tan\phi\tan\theta = 1$

Also using $v^2 = u^2 + 2as$ in the x direction $0 = V^2\cos^2\theta - 2gR\sin\phi$

i.e. $R = \frac{V^2\cos^2\theta}{2g\sin\phi}$.

Thus

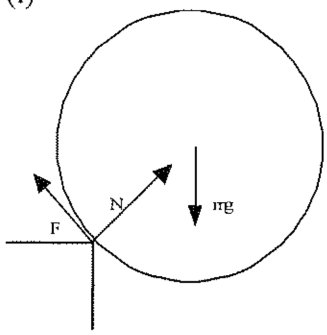
$\frac{2V^2}{gR} = 4\sin\phi\sec^2\theta = 4\sin\phi(1 + \tan^2\theta) = 4\sin\phi\left(1 + \frac{1}{4}\cot^2\phi\right) = 4\sin\phi\left(1 + \frac{1}{4}(\mathrm{cosec}^2\phi - 1)\right)$

$= 3\sin\phi + \mathrm{cosec}\phi$

Consider $y = 3x + \frac{1}{x}, x > 0$. By differentiation, this is least for $x = \frac{1}{\sqrt{3}}$.

Thus the least value of $\frac{2V^2}{gR}$ is $2\sqrt{3}$, and the largest value of R is $\frac{V^2}{\sqrt{3}g}$.

11. (i)



If the angle between mg and N is θ, then conserving energy and either differentiating the energy equation or taking moments about the point of contact yields

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}ma^2\dot{\theta}^2 + mga\cos\theta \text{ and } 0 = a\ddot{\theta} - g\sin\theta$$

Resolving in the opposite direction to F, $mg\sin\theta - F = ma\ddot{\theta}$ and so, from the second equation above, $F = 0$.

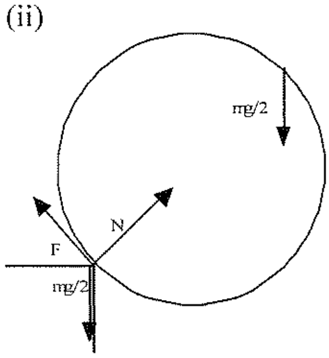
Resolving in the opposite direction to N, $mg\cos\theta - N = ma\dot{\theta}^2$,

and losing contact $N = 0$, so $a\dot{\theta}^2 = g\cos\theta$.

Thus from the energy equation $u^2 + 2ag = 3ag\cos\theta$ and so the hub has fallen

$$a - a\cos\theta = a - \frac{u^2 + 2ag}{3g} = \frac{ag - u^2}{3g} > 0, \text{ but is less than } a.$$

(ii)



As before $\frac{1}{2}\frac{m}{2}(2u)^2 + \frac{m}{2}g(2a) = \frac{1}{2}\frac{m}{2}(2a)^2\dot{\theta}^2 + \frac{m}{2}g(2a)\cos\theta$ and $0 = 2a\ddot{\theta} - g\sin\theta$,

and $mg\sin\theta - F = \frac{m}{2}(2a)\ddot{\theta}$ so $F = \frac{1}{2}mg\sin\theta$.

Also $mg\cos\theta - N = \frac{m}{2}(2a)\dot{\theta}^2$ and so when contact is lost $N = 0$, so $a\dot{\theta}^2 = g\cos\theta$,

$u^2 + ag = 2ag\cos\theta$,

and the hub has fallen $a - a\cos\theta = a - \dfrac{u^2 + ag}{2g} = \dfrac{ag - u^2}{2g} > 0$, but is less than a.

So when $N = 0$, $\mu N = 0$, $F > 0$, but we require $F < \mu N$ not to slip, and hence slipping will certainly occur before it loses contact with the table.

Section C: Probability and Statistics

12.

$$E(N) = \sum_{i=1}^{2n-1} \frac{1}{2n-1} i = \frac{1}{2n-1} \frac{(2n-1)2n}{2} = n$$

$$E(N^2) = \sum_{i=1}^{2n-1} \frac{1}{2n-1} i^2 = \frac{1}{2n-1} \frac{(2n-1)2n(4n-1)}{6} = \frac{n(4n-1)}{3}$$

$$E(Y) = E\left(\sum_{i=1}^{N} X_i\right) = \frac{1}{2n-1} E(X_1) + \frac{1}{2n-1} E(X_1 + X_2) + \ldots = \frac{1}{2n-1}(\mu + 2\mu + 3\mu + \ldots + (2n-1)\mu)$$

$$= \frac{1}{2n-1} \frac{\mu(2n-1)2n}{2} = n\mu$$

$$E(YN) = \frac{1}{2n-1} \times 1 \times \mu + \frac{1}{2n-1} \times 2 \times 2\mu + \ldots + \frac{1}{2n-1} \times (2n-1) \times (2n-1)\mu = \frac{n(4n-1)}{3} \mu$$

and so $Cov(Y, N) = \frac{n(4n-1)}{3} \mu - n^2 \mu = \frac{1}{3} n(n-1)\mu$

$$E\left(X_i^{\,2}\right) = Var(X_i) + \left(E(X_i)\right)^2 = \sigma^2 + \mu^2$$

Also $\left(X_1 + X_2 + \ldots + X_r\right)^2 = \sum_{i=1}^{r} X_i^{\,2} + 2\sum_{i \neq j} X_i X_j$, and so

$$E\left(\left(X_1 + X_2 + \ldots + X_r\right)^2\right) = r\left(\sigma^2 + \mu^2\right) + 2\frac{r(r-1)}{2}\mu^2$$

Thus

$$E(Y^2) = \frac{1}{2n-1} \sum_{r=1}^{2n-1} \left(r\left(\sigma^2 + \mu^2\right) + 2\frac{r(r-1)}{2}\mu^2 \right) = n\left(\sigma^2 + \mu^2\right) + \frac{n(4n-1)}{3}\mu^2 - n\mu^2 = n\sigma^2 + \frac{n(4n-1)}{3}\mu^2$$

and so $Var(Y) = n\sigma^2 + \frac{n(4n-1)}{3}\mu^2 - n^2\mu^2 = n\sigma^2 + \frac{n(n-1)}{3}\mu^2$

13. (i) $p_2(2)$ is the probability of landing in the pool for the first time on the 2nd jump starting 1.5m away which is the probability that the first jump is 1m which is p.

(ii) $u_1 = 1$

$p_2(1) = q$ and $p_2(2) = p$ so $u_2 = q + 2p = 1 + p = 2 - q$

$p_3(1) = 0$, $p_3(2) = 1 - p^2 = q(1+p) = 2q - q^2$, and $p_3(3) = p^2 = 1 - 2q + q^2$ so

$u_3 = 2\left(2q - q^2\right) + 3\left(1 - 2q + q^2\right) = 3 - 2q + q^2$

(iii) Using the values $u_1 = 1$, $u_2 = 2 - q$, and $u_3 = 3 - 2q + q^2$, we obtain three equations:-

$A + B + C = 1$ (1)

$-Aq + B + 2C = 2 - q$ (2)

$Aq^2 + B + 3C = 3 - 2q + q^2 \quad (3)$

It makes sense to consider $(3)-(2)$ and $(2)-(1)$ to eliminate B and then subtract the resulting equations to eliminate C, and hence we find that

$(3) - 2(2) + (1) \Rightarrow A\left(q^2 + 2q + 1\right) = q^2 \Rightarrow A = \left(\dfrac{q}{q+1}\right)^2$,

substituting in $(2)-(1) \Rightarrow \left(\dfrac{q}{q+1}\right)^2 (-q-1) + C = 1 - q \Rightarrow C = \dfrac{1}{1+q}$, and so

$B = \dfrac{q}{(q+1)^2}$.

So $u_n = \left(\dfrac{q}{q+1}\right)^2 (-q)^{n-1} + \dfrac{q}{(q+1)^2} + \dfrac{1}{1+q} n = \dfrac{(-q)^{n+1}}{(q+1)^2} + \dfrac{q}{(q+1)(p+2q)} + \dfrac{1}{p+2q} n$

For large n, the first term approaches zero, and the second term is negligible in comparison with the third for $\dfrac{q}{q+1} < 1 << n$

Hence $u_n \approx \dfrac{1}{p+2q} n$

The expected distance covered in one jump is $q + 2p$ and as jumps are of integer length, to get to the pool from a distance $\left(n - \dfrac{1}{2}\right)m$ needs a distance n metres to be jumped and so the expected number of jumps would be $\dfrac{1}{p+2q} n$.

14. (i) If W is the area of the smallest circle with centre O that encloses the hole made by a single dart throw then the p.d.f. of W is given by

$$f(w) = \begin{cases} \dfrac{1}{\pi}, 0 \le w \le \pi \\ 0, otherwise \end{cases}$$

If X is the area of the smallest circle with centre O that encloses all the n holes made then

$P(x < X < x + \delta x) = n\left(\dfrac{x}{\pi}\right)^{n-1} \dfrac{\delta x}{\pi}$ and so $E(X) = \displaystyle\int_0^\pi x \times n\left(\frac{x}{\pi}\right)^{n-1} \frac{1}{\pi} dx = \frac{n\pi}{n+1}$.

On the other hand, if Y is the area of the smallest circle with centre O that encloses all the $(n-1)$ holes nearest to O then $P(x < Y < x + \delta x) = n(n-1)\left(\dfrac{x}{\pi}\right)^{n-2}\left(1 - \dfrac{x}{\pi}\right)\dfrac{\delta x}{\pi}$ and

so $E(Y) = \displaystyle\int_0^\pi x \times n(n-1)\left(\left(\frac{x}{\pi}\right)^{n-2} - \left(\frac{x}{\pi}\right)^{n-1}\right)\frac{1}{\pi} dx = \frac{(n-1)\pi}{n+1}$

(ii) If Z is the area of the smallest square with centre Q that encloses all the n holes made then, in similar manner to (i)

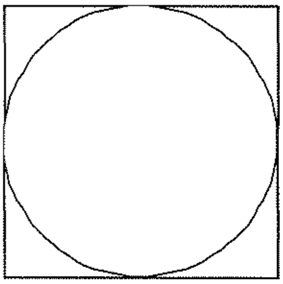
$P(x < Z < x + \delta x) = n\left(\dfrac{x}{4}\right)^{n-1} \dfrac{\delta x}{4}$ and so $E(Z) = \displaystyle\int_0^4 x \times n\left(\frac{x}{4}\right)^{n-1} \frac{1}{4} dx = \frac{4n}{n+1}$.

(iii) If we knew that the dart landed inside the circle of radius 1 centre Q when it hit the square dartboard, then the answer would be that we obtained for the circular board. But there is a non-zero probability that the dart could land in larger circles if it fell on the board outside the circle of radius 1 and hence the expected area of the smallest circle for the square dartboard is larger than that for the circular board.

Algebraically, if S is the expected area of such a circle if the dart falls outside the circle on the square board, and *E(X)* is as in part (i),

the expected area = $\left(\frac{\pi}{4}\right)E(X)+\left(1-\frac{\pi}{4}\right)S$, where $S > E(X)$, and so this is

$$\left(1-\left(1-\frac{\pi}{4}\right)\right)E(X)+\left(1-\frac{\pi}{4}\right)S = E(X)+\left(1-\frac{\pi}{4}\right)(S-E(X)) > E(X)$$



STEP Mathematics (9465, 9470, 9475)

Report on the Components

June 2007

Unit	**Content**
9465	STEP Mathematics I
9470	STEP Mathematics II
9475	STEP Mathematics III
	Unit Threshold Marks

STEP Mathematics I 2007: Report

General comments

There were significantly more candidates attempting this paper this year (an increase of nearly 50%), but many found it to be very difficult and only achieved low scores. In particular, the level of algebraic skill required by the questions was often lacking. The examiners' express their concern that this was the case despite a conscious effort to make the paper more accessible than last year's. At this level, the fluent, confident and correct handling of mathematical symbols (and numbers) is necessary and is expected; many good starts to questions soon became unstuck after a simple slip. Graph sketching was usually poor: if future candidates wanted to improve one particular skill, they would be well advised to develop this.

There were of course some excellent scripts, full of logical clarity and perceptive insight. It was pleasing to note that the applied questions were more popular this year, and many candidates scored well on at least one of these. It was however surprising how rarely answers to questions such as 5, 9, 10, 11 and 12 began with a diagram.

However, the examiners were left with the overall feeling that some candidates had not prepared themselves well for the examination. The use of past papers to ensure adequate preparation is strongly recommended. A student's first exposure to STEP questions can be a daunting, demanding experience; it is a shame if that takes place during a public examination on which so much rides.

Further, and fuller, discussion of the solutions to these questions can be found in the *Hints and Answers* document.

Comments on specific questions

1 This question required little more than a clear head and some persistence: candidates had either ample or very little of both, and thus most scores were either high or very low. The examiners would like to stress that a solution to a question such as this must be written out methodically and coherently: many answers which began promisingly were soon hopelessly fragmented and incoherent, leaving the candidate unable to regain his or her train of thought. This was especially true when deriving the final expression given on the exam paper. Examiners follow closely a candidate's line of reasoning, and they have to be certain that the candidate has constructed a complete argument, and that he or she has not arrived at a printed result without full justification.

2 This was a popular question, and was usually well done. The argument at the end was often incomplete, though: many candidates simply stated that $t = 1$ or $t = 2$ without explaining why no other values were possible. To do so, use had to be made of the fact that s and t have no common factor other than 1.

3 This was the most popular question on the paper, and many different methods were seen. The intended method was to use the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$ to evaluate the integrals of $\cos^4\theta - \sin^4\theta$ and $\cos^4\theta + \sin^4\theta$, and hence be able to write down

separately the values of the integrals of $\cos^4 \theta$ and $\sin^4 \theta$. A similar approach works well for $\cos^6 \theta - \sin^6 \theta$ and $\cos^6 \theta + \sin^6 \theta$. Other methods were, of course, acceptable, and many candidates received high marks for this question.

4 This question was found to be very difficult. The initial factorisation was beyond most candidates, even given the linear factor $x + b + c$. Anyone who wants to read Mathematics at university must be able to factorise quickly cubic expressions such as this one, and also $x^3 \pm y^3$. The *Hints and Answers* document discusses this in more detail.

Candidates who progressed to the second part of the question often deduced that $ak^2 + bk + c = 0$ and $bk^2 + ck + a = 0$, but then tried to eliminate k; given that the result they were asked to derive was still in terms of k, this was an unwise strategy.

5 Only a few candidates made much progress with this question, even though it only required GCSE Mathematics. Basic properties of triangles (for example, the sine and cosine rule, and the location of the centroid, the circumcentre and the incentre) are assumed knowledge at this level. It was surprising how many candidates tried to answer this question without a diagram.

6 This was a popular, straightforward question, which was often answered well. However, algebraic errors still occurred, for example when expanding $(x - y)^3$.

7 Part (i) was well done by most of those who attempted this question, but many then found it difficult to develop the strategy in part (ii). A certain amount of trial and error is needed to complete the squares in an expression in terms of both α and β, but the coefficients (in particular, $\mathbf{1}\alpha^2$, $\mathbf{1}\beta^2$ and $\mathbf{26}\beta^2 k^2$) do not permit many possibilities. This question demanded some stamina, as Mathematics at university level also does.

8 This question was answered poorly; many candidates were unable to sketch the graphs correctly, even given the results derived earlier in the question. For example, many graphs did not touch at (2, 8). Also, many graphs were drawn with turning points, when a simple check of the derivative would have revealed that there were none. In part (iii), the effect of the negative coefficient of x^3 was often ignored.

Graph sketching is a very important skill in all mathematical subjects – from Economics to Engineering. STEP candidates are strongly advised to practise this skill as much as possible.

9 This was a popular question, and was usually well done. Not many candidates recognised that $\sin \theta \, \cos \theta \equiv \frac{1}{2} \sin 2\theta$, which makes the final inequality easier to obtain. Knowing identities "both ways" is important.

10 Only a few attempts at this question were seen, and those that did rarely made much headway; worryingly, the accurate simplification of the solutions of simple linear equations was found to be very difficult.

11 Hardly any attempts at this question were seen. It was remarkable how few diagrams were seen; it is always much easier for both the candidate and the examiner if answers begin with a labelled diagram.

12 Very few tree diagrams were seen here, and hence very few correct solutions were constructed; a clear tree diagram is invaluable when attempting a complicated probability question such as part (ii). Most candidates identified some (if not all) of the possible outcomes, but many mistakes were made (for example, writing a denominator of N rather than $N + 1$ or $N - 1$).

The subsequent algebraic simplification was found to be very demanding. Candidates would have probably made more progress if they had been more willing to factorise groups of terms which had obvious common factors, rather than (for example) attempting to write all the fractions with a common denominator.

13 A lot of attempts at this question were seen, but conceptual errors undermined many solutions. In particular, a lot of candidates seemed not to realise that they were being asked to calculate conditional probabilities in parts (ii) to (vi).

14 Only a few attempts at this question were seen. Poor graph sketching limited many candidates' progress; the importance of the ability to sketch accurately standard graphs such as $y = xe^{-x}$ cannot be overstated.

STEP Mathematics II 2007: Report

General Remarks

Although the paper was by no means an easy one, it was generally found a more accessible paper than last year's, with most questions clearly offering candidates an attackable starting-point. The candidature represented the usual range of mathematical talents, with a pleasingly high number of truly outstanding students; many more who were able to demonstrate a thorough grasp of the material in at least three questions; and the few whose three-hour long experience was unlikely to have been a particularly pleasant one. However, even for these candidates, many were able to make some progress on at least two of the questions chosen.

Really able candidates generally produced solid attempts at five or six questions, and quite a few produced outstanding efforts at up to eight questions. In general, it would be best if centres persuaded candidates not to spend valuable time needlessly in this way – it is a practice that is not to be encouraged, as it uses valuable examination time to little or no avail. Weaker brethren were often to be found scratching around at bits and pieces of several questions, with little of substance being produced on more than a couple. It is an important examination skill – now more so than ever, with most candidates now not having to employ such a skill on the modular papers which constitute the bulk of their examination experience – for candidates to spend a few minutes at some stage of the examination deciding upon their optimal selection of questions to attempt.

As a rule, question 1 is intended to be accessible to all takers, with question 2 usually similarly constructed. In the event, at least one – and usually both – of these two questions were among candidates' chosen questions. These, along with questions 3 and 6, were by far the most popularly chosen questions to attempt. The majority of candidates only attempted questions in Section A (Pure Maths), and there were relatively few attempts at the Applied Maths questions in Sections B & C, with Mechanics proving the more popular of the two options.

It struck me that, generally, the working produced on the scripts this year was rather better set-out, with a greater logical coherence to it, and this certainly helps the markers identify what each candidate thinks they are doing. Sadly, this general remark doesn't apply to the working produced on the Mechanics questions, such as they were. As last year, the presentation was usually appalling, with poorly labelled diagrams, often with forces missing from them altogether, and little or no attempt to state the principles that the candidates were attempting to apply.

Comments on responses to individual questions

SECTION A: PURE MATHEMATICS

Q1 Most candidates attempted this question and the majority coped fairly well with the algebraic demands. Surprisingly, it was when the work went numerical that candidates tended to let themselves down; poor arithmetic providing the main difficulty. The final three marks available in (i) parts (a) and (b) were the marks most frequently scorned, generally being lost by candidates' unwillingness or inability to simplify fractions and/or turn them into decimals. In many cases, candidates had difficulty deciding on a suitable value for k in (i) (b) and (ii). In (b), the value $k = 50$ was often selected, rather than the intended value of -4. Although this does lead to a similar set of working, the ultimate approximation is relatively poor, and they lost the final mark here. It is, rather more tellingly, indicative of the way in which many modern A-level mathematicians have great difficulty in thinking only in terms of positive integers! A small, but significant, number of candidates offered a value of k that exceeded 100 (the denominator of the "x" term), and these were penalised all four of the available marks for this part of the question, on the not unreasonable grounds that they really should have appreciated the general convergence condition $\left| \frac{k}{100} \right| < 1$ for binomial series of this kind.

Q2 This question was also a very popular one, although many candidates gave up their attempt when the algebra started to get a little too tough for them, which generally happened later if not sooner. With this in mind, it has to be said that when candidates *did* get stuck at some stage of this question, the principal cause was (again!) an unwillingness or inability to simplify algebraic expressions before attempting to work with them. This was particularly important when factorising otherwise lengthy expressions with lots of $(q - p)$s involved in them.

The sketch required in (ii) was intended to be a gentle prod in the right direction for later use in the question, and should have been four easy marks for the taking. Strangely, however, it was often not very well attempted at all. A surprising number of candidates couldn't even manage to draw their cubic through O; and many others seemed unable to make good use of the given conditions, which – despite looking complicated – actually just ensured that all the fun was going on in the first quadrant in an attempt to make life easy. Even more surprising still was the number of sketches that had non-cubic-like kinks, bumps and extra inflection points in them. I was particularly baffled by this widespread lack of grasp as to what a cubic should actually look like! I was equally baffled by the extraordinarily large body of candidates who failed to do what the question explicitly told them they were required to do, by not marking the point of inflection on their sketch and, in many cases, not even attempting to describe the symmetry of it either.

An apology has to be made at this point, since the region $\boldsymbol{R}$ in the question was insufficiently clearly defined and there were, in fact, two possibilities. Candidates were not penalised for choosing the "wrong" one at any stage of the proceedings, although the choice of the "left-hand" $\boldsymbol{R}$ would have prevented such candidates from using the short-cut for the following attempt at the area. ALL scripts where candidates made the "wrong" choice were passed to the Principal Examiner and given careful individual consideration. Only about 25 candidates made such a choice: of these, over half had failed to make any attempt at all at the area, and most of the rest had started work on the area and, to all intents and purposes, given up immediately. Two more had found the intended area anyway, despite their previous working (and were not penalised for having switched regions), and (I think) only three had pursued the "left-hand" area almost to a conclusion. Of course, they were unable to get the given answer, but they did get 7 of the 8 marks available. In each of these cases, it was fortunate (for us and them) that this was their last question, so it was safe to say that they hadn't been unduly penalised for time in any way. It is, of course, impossible to say whether they might have seen the intended short-cut approach. In this respect, however, it has to be said that remarkably few candidates saw the symmetry approach anyhow. Partly, I suspect, due to not having picked up the hint at the diagram stage (see earlier)! On the plus side, for us, I imagine that the reference to the point of inflection on the diagram had at least ensured that most candidates chose the intended region $\boldsymbol{R}$. Only 2 of the 25 or so candidates scored an overall mark that fell just below a grade boundary, and both of these were given the benefit of the doubt by the Chief Examiner.

Q3 This was another popular question, and was usually a good source of marks for those candidates who attempted it. The first two parts were usually successfully completed. In part (i) (b), candidates had to employ the $t = \tan \frac{1}{2} x$ substitution which seems to have fallen into disuse in recent years (due to modularity!). Having said that, most candidates were able to make some progress and, where they did fall down, it was generally due to a lack of confidence in handling trigonometric identities. One of the advantages of these last two parts to the question was that they could be done in one of two directions, and many candidates were able to spot the connections and exploit them satisfactorily. When errors arose, they were frequently due to a lack of care with constants, and a correct final answer was not often to be found as a result.

Q4 This question was a popular one for partial attempts; with most candidates giving up towards the end of the introductory part and going elsewhere. It was slightly surprising to see candidates being put off in this way, since the given result made it perfectly possible to move successfully into the three following cases. For those who did press on, many lost a mark for not verifying (somewhere) that the chosen values of α, β and γ actually satisfied the required condition. Then, in (ii), one of the two brackets was identically zero, the significance of which was largely overlooked, with many candidates offering again the same two solutions as had been found in part (i). In (iii), it was important to note first A and B in terms of x, although some candidates adopted a valid alternative approach by first collecting up the two $3x$ terms.

Q5 Although this was not a popular choice of question, those who attempted it generally did rather well on it. Finding f^2 and f^3 was a routine algebraic slog, and most attempts coped successfully with it. Spotting, and then exploiting, the periodicity of the function was then a relatively easy matter. Pretty much everyone used $x = \tan \theta$ appropriately in (ii), with formal and informal induction approaches evenly mixed. Some shrewder candidates identified the two forms for the cases $n = 1$, 2, and 3 and then noted that the periodicity of the tan function accounted for everything thereafter.

The final part of the question had intended to be a simple take on part (ii), but with $t = \sin \theta$ this time, so that $\sqrt{1-t^2} = \cos \theta$, and attempts at this part of the question generally fell evenly into one of the two following camps: those who gave up, and those who proceeded as intended. In all, I think there were just three candidates who noticed the extra complication that can arise in this case, with just two or three more following a separate line of enquiry without realising the inherent dichotomy in the "powers" of the function g. A full inspection of the function exposes the fact that g^n takes different forms depending upon which part of the domain of g is employed. This is because the $\sqrt{1-t^2}$ bit should actually be $|\cos \theta|$, and this leads to different answers for g^2 in the range $\frac{1}{2} \le t \le 1$ than in the rest of g's domain, so that candidates could get different answers from slightly different approaches. With so few candidates expected to attempt this last part of the question, and with the alternate route leading to a much easier answer (where the sequence g^n turns out to be periodic with period 2), it was considered to be a suitable final part to the question. Candidates were not expected to take more than one route, nor to comment on the potential for different answers. In the event, none did the former, although a few gave a mention of the latter property.

Q6 This was one of the most popular questions on the paper, although the number of completely successful attempts could be counted without having to resort to toes! Part (i) was reasonably routine, although attempts at simplification were often not very well done, and left many candidates having to resort to "otherwise" approaches for integrating $\sqrt{3+x^2}$, which was a great shame as they got no marks for ignoring the "hence" instruction in the question. Treating the differential equation in part (ii) as a quadratic in dy/dx proved an obstacle for many, but a lot of candidates seemed quite happy to work with it as such and made good progress in the rest of the question. The biggest hurdle to completely successful progress, however, once again lay in candidates' inability to simplify expressions at various stages, and sign and/or constant errors proliferated.

Q7 Not very many candidates attempted this question, but those who did usually found it to be relatively straightforward. It was only the very last part that required much thought, and this was where most attempts lost a few marks. A small number of efforts failed to get beyond part (ii); this was due to not finding a suitable function to work with that gave what turns out to be the *Arithmetic Mean-Geometric Mean Inequality*. This was a bit of a shame, since the question actually gives the log. function at the very beginning, along with the sine function, which is used in (i).

Q8 This is really just half of the ($\Leftrightarrow$) proof of Ceva's Theorem. Several candidates even recognised it as such. Of the remarkably small number of attempts submitted, most fell down at some stage (again) by failing to be sufficiently careful with signs/arithmetic/the modest amounts of algebra involved. It often didn't help those candidates who chose completely different symbols each time they did a stage of the working.

SECTION B: MECHANICS

Q9 This was the least popular of the Mechanics questions, perhaps because it commenced with a request for an explanation. As mentioned already, a clearly labelled diagram or two would have been enormously helpful here! The fact that there are only the two mechanical principles being employed here should have made it an easy question, but efforst were generally very poor.

Q10 This was most popular of the three Mechanics questions, although most efforts failed to get very far into it. The routine opening part, finding the position of a centre of mass, probably accounts for its initial (relative) popularity, but progress beyond this point was pitifully weak in most cases. Resolving and taking moments frequently appeared, but often had to be searched-for in amidst a sea of other statements, many of which were incorrect, repetitive or just nonsensical. Very few candidates indeed grasped the fact that the horizontal force P could be in either direction, and the given answer was mostly fiddled, usually by simply placing modulus signs around the answer.

Q11 This was almost as popular a question on Section B as Q10, being (in principle, at least) a reasonably straightforward projectiles question. Whilst many efforts were successful up to the final part, an awful lot of the attempts foundered at the very outset by failing to do the simplest of tasks: namely, noting exact values for $\sin\theta$ and $\cos\theta$ from $\tan\theta = \frac{1}{2}$. It simply beggars belief that serious candidates can proceed through quite a large part of a question like this with expressions such as $\sin(\arctan \frac{1}{2})$ still in there! They may as well just hang out a flag which says "I'm an incompetent mathematician" on it! The three-dimensional aspect of the introduction was enough to confound most candidates attempting this question, and they were forced to resort to fiddling the given answer for the distance OP. Many attempts picked up several marks here and there throughout the question without producing anything particularly coherent, and few coped with the hazards of the last part – largely, I suspect, due to the fact that they were required to do some approximating!

SECTION C: STATISTICS

Q12 This was the least popular of the Statistics questions, with very few attempts seen on it. Most of these tended to consist of muddled or unexplained reasoning which led to the fiddling of (i)'s given result. Progress into (ii) was either non-existent or sketchy as a consequence.

Q13 This is a lovely approach to a well-known problem, and employs a very handy rational approximation to ln 2. Although it drew a small number of attempts, many of these were partial attempts at best, and few were seen of a good standard throughout. Disappointingly, several candidates arriving at the correct quadratic equation in the third part didn't seem to know how to go about solving it. As was mentioned earlier, regarding the end of Q11, working with approximations proved to be a particular obstacle for most candidates who made it to the last part here.

Q14 This was the most popular of the Statistics questions, probably due to the high pure mathematical content. The sketch introduction was intended to ensure that candidates drew something which would remind them what integrals they should be working with later on. As with Q2, it presented more problems than should have been the case, with many candidates losing marks for fairly trivial things which would have cost them dearly even on an ordinary AS/A-level module paper. The integration for total probability was generally done very well, although several candidates had often failed to gain a and b in terms of k in a simplified form, or at all, and this rather hindered them. In (iii), most candidates didn't seem to feel that it was necessary to justify which region of the function that the median lay in, often doing one calculation after making an assumption about the matter. In general, it is always best if candidates can justify their choices.

STEP Mathematics III 2007: Report

Section A: Pure Mathematics

1. This question was popular. Many candidates did not simplify their first expression into the symmetrical form which made it harder for them to spot the use of the sums and products of roots results. A common slip was to make a 1 by default which also obscured what was going on. Most struggled to take the given equation requiring solution and produce a quartic equation in t ($\tan\vartheta$), some producing a quartic equation in $\cos\vartheta$, and somehow expecting to use the earlier results.

2. This question was popular though not well answered. Solutions to part (i) were frequently unconvincing, though to part (ii) were quite good if they avoided elementary errors in working. Part (iii) was less well attempted with some not spotting to use integration, some stumbling over "+ c" and some not spotting the value of x to substitute.

3. This question was popular. Many solutions to part (ii) were rambling and lacked a sense of direction, even if correct. The induction in (iii) was frequently incorrectly handled and a common error was to replace n by $k/2$. Part (iv) caused difficulties.

4. This question was quite popular. A lot of attempts involved rambling trigonometrical manipulations, and few spotted the standard differential of $\ln\tan\frac{t}{2}$. The curve sketch was often omitted or incorrect, and there was a lot of complicated working using e.g. the equation of the normal etc. to find the centre of curvature.

5. This was frequently attempted, though lack of facility with hyperbolic functions meant that few progressed beyond the first two differentials, and for those going further, the working was not methodical enough to spot the factorial that would emerge in the general result.

6. This was the least popular Pure question and very little success was achieved by the few that attempted it. The first result was often obtained correctly by expressing each of the four complex numbers in modulus-exponential form, but then the perpendicularity was the stumbling block.

7. This was a very popular question. As the question led the candidates through there were a number of unconvincing solutions to parts of the question, but overall it was reasonably well handled.

8. This ranked alongside question 5 in popularity and success. Frequently, it was calculation errors that obscured the path through part (i) and the two differences between part (i) and part (ii) were enough to put most off the track for part (ii), even if they had completed (i) successfully.

Section B: Mechanics

9. This was little attempted. Some did struggle through to the solution of the differential equation, but the appreciation of the three possible cases eluded them.

10. This was the most popular of the Mechanics questions, but less so than any but question 6 of the Pure. Most managed to obtain the first two results correctly, but then struggled to find the further result. The deduction for the largest R was rarely spotted leading to some unnecessarily unwieldy calculus.

11. There were very few attempts at this question.

Section C: Probability and Statistics

12. There were some attempts at this question but they faltered when trying to find the expectation of Y, even though some may have believed that they had obtained the required result through false logic.

13. This was the most popular of the Probability and Statistics questions, ranking alongside questions 5 and 8. The first two parts were competently handled, but most got bogged down in the algebra of part (iii) through not having a clear strategy to solve the equations.

14. There were few answers of any substance to this question.

STEP Mathematics (9465, 9470, 9475)
June 2007 Assessment Series

Unit Threshold Marks

Unit	Maximum Mark	S	1	2	3	U
9465	120	81	66	49	36	0
9470	120	95	67	56	35	0
9475	120	86	64	52	35	0

The cumulative percentage of candidates achieving each grade was as follows:

Unit	S	1	2	3	4
9465	7.08	16.46	45.84	69.74	100
9470	12.89	38.10	53.50	83.19	100
9475	12.63	38.30	57.56	85.09	100

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