

## Section A: Pure Mathematics

**1** Let  $x = 10^{100}$ ,  $y = 10^x$ ,  $z = 10^y$ , and let

$$a_1 = x!, \quad a_2 = x^y, \quad a_3 = y^x, \quad a_4 = z^x, \quad a_5 = e^{xyz}, \quad a_6 = z^{1/y}, \quad a_7 = y^{z/x}.$$

- (i) Use Stirling's approximation  $n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$ , which is valid for large  $n$ , to show that  $\log_{10}(\log_{10} a_1) \approx 102$ .
- (ii) Arrange the seven numbers  $a_1, \dots, a_7$  in ascending order of magnitude, justifying your result.

**2** Consider the quadratic equation

$$nx^2 + 2x\sqrt{(pn^2 + q)} + rn + s = 0, \quad (*)$$

where  $p > 0$ ,  $p \neq r$  and  $n = 1, 2, 3, \dots$ .

- (i) For the case where  $p = 3$ ,  $q = 50$ ,  $r = 2$ ,  $s = 15$ , find the set of values of  $n$  for which equation (\*) has no real roots.
- (ii) Prove that if  $p < r$  and  $4q(p - r) > s^2$ , then (\*) has no real roots for any value of  $n$ .
- (iii) If  $n = 1$ ,  $p - r = 1$  and  $q = s^2/8$ , show that (\*) has real roots if, and only if,  $s \leq 4 - 2\sqrt{2}$  or  $s \geq 4 + 2\sqrt{2}$ .

**3** Let

$$S_n(x) = e^{x^3} \frac{d^n}{dx^n} (e^{-x^3}).$$

Show that  $S_2(x) = 9x^4 - 6x$  and find  $S_3(x)$ .

Prove by induction on  $n$  that  $S_n(x)$  is a polynomial. By means of your induction argument, determine the order of this polynomial and the coefficient of the highest power of  $x$ .

Show also that if  $\frac{dS_n}{dx} = 0$  for some value  $a$  of  $x$ , then  $S_n(a)S_{n+1}(a) \leq 0$ .

- 4 By considering the expansions in powers of  $x$  of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^n \binom{n}{s}^2 = \binom{2n}{n},$$

where  $\binom{n}{s} = \frac{n!}{s!(n-s)!}$ .

By considering similar identities, or otherwise, show also that:

- (i) if  $n$  is an even integer, then

$$\sum_{s=0}^n (-1)^s \binom{n}{s}^2 = (-1)^{n/2} \binom{n}{n/2};$$

(ii) 
$$\sum_{t=1}^n 2t \binom{n}{t}^2 = n \binom{2n}{n}.$$

- 5 Show that if  $\alpha$  is a solution of the equation

$$5\cos x + 12\sin x = 7,$$

then either

$$\cos \alpha = \frac{35 - 12\sqrt{120}}{169}$$

or  $\cos \alpha$  has one other value which you should find.

Prove carefully that if  $\pi/2 < \alpha < \pi$ , then  $\alpha < 3\pi/4$ .

- 6 Find  $\frac{dy}{dx}$  if

$$y = \frac{ax + b}{cx + d}. \quad (*)$$

By using changes of variable of the form  $(*)$ , or otherwise, show that

$$\int_0^1 \frac{1}{(x+3)^2} \ln \left( \frac{x+1}{x+3} \right) dx = \frac{1}{6} \ln 3 - \frac{1}{4} \ln 2 - \frac{1}{12},$$

and evaluate the integrals

$$\int_0^1 \frac{1}{(x+3)^2} \ln \left( \frac{x^2 + 3x + 2}{(x+3)^2} \right) dx \quad \text{and} \quad \int_0^1 \frac{1}{(x+3)^2} \ln \left( \frac{x+1}{x+2} \right) dx.$$

- 7 The curve  $C$  has equation

$$y = \frac{x}{\sqrt{(x^2 - 2x + a)}},$$

where the square root is positive. Show that, if  $a > 1$ , then  $C$  has exactly one stationary point.

Sketch  $C$  when (i)  $a = 2$  and (ii)  $a = 1$ .

- 8 Prove that

$$\sum_{k=0}^n \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta}. \quad (*)$$

- (i) Deduce that, when  $n$  is large,

$$\sum_{k=0}^n \sin \left( \frac{k\pi}{n} \right) \approx \frac{2n}{\pi}.$$

- (ii) By differentiating (\*) with respect to  $\theta$ , or otherwise, show that, when  $n$  is large,

$$\sum_{k=0}^n k \sin^2 \left( \frac{k\pi}{2n} \right) \approx \left( \frac{1}{4} + \frac{1}{\pi^2} \right) n^2.$$

[The approximations, valid for small  $\theta$ ,  $\sin \theta \approx \theta$  and  $\cos \theta \approx 1 - \frac{1}{2}\theta^2$  may be assumed.]

## Section B: Mechanics

**9** In the  $Z$ -universe, a star of mass  $M$  suddenly blows up, and the fragments, with various initial speeds, start to move away from the centre of mass  $G$  which may be regarded as a fixed point. In the subsequent motion the acceleration of each fragment is directed towards  $G$ . Moreover, in accordance with the laws of physics of the  $Z$ -universe, there are positive constants  $k_1$ ,  $k_2$  and  $R$  such that when a fragment is at a distance  $x$  from  $G$ , the magnitude of its acceleration is  $k_1x^3$  if  $x < R$  and is  $k_2x^{-4}$  if  $x \geq R$ . The initial speed of a fragment is denoted by  $u$ .

- (i) For  $x < R$ , write down a differential equation for the speed  $v$ , and hence determine  $v$  in terms of  $u$ ,  $k_1$  and  $x$  for  $x < R$ .
- (ii) Show that if  $u < a$ , where  $2a^2 = k_1R^4$ , then the fragment does not reach a distance  $R$  from  $G$ .
- (iii) Show that if  $u \geq b$ , where  $6b^2 = 3k_1R^4 + 4k_2/R^3$ , then from the moment of the explosion the fragment is always moving away from  $G$ .
- (iv) If  $a < u < b$ , determine in terms of  $k_2$ ,  $b$  and  $u$  the maximum distance from  $G$  attained by the fragment.

**10**  $N$  particles  $P_1, P_2, P_3, \dots, P_N$  with masses  $m, qm, q^2m, \dots, q^{N-1}m$ , respectively, are at rest at distinct points along a straight line in gravity-free space. The particle  $P_1$  is set in motion towards  $P_2$  with velocity  $V$  and in every subsequent impact the coefficient of restitution is  $e$ , where  $0 < e < 1$ . Show that after the first impact the velocities of  $P_1$  and  $P_2$  are

$$\left(\frac{1 - eq}{1 + q}\right)V \quad \text{and} \quad \left(\frac{1 + e}{1 + q}\right)V,$$

respectively.

Show that if  $q \leq e$ , then there are exactly  $N - 1$  impacts and that if  $q = e$ , then the total loss of kinetic energy after all impacts have occurred is equal to

$$\frac{1}{2}me(1 - e^{N-1})V^2.$$

- 11 An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius  $R$  in a horizontal plane at a constant angular speed  $\omega$ . A shell is fired from  $O$ , the centre of this circle, with initial speed  $V$  and angle of elevation  $\alpha$ . Show that if  $V^2 < gR$ , then no matter what the value of  $\alpha$ , or what vertical plane the shell is fired in, the shell cannot hit the target.

Assume now that  $V^2 > gR$  and that the shell hits the target, and let  $\beta$  be the angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that  $\beta$  satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of  $\beta$ .

Let  $\beta_1$  and  $\beta_2$  be the possible values of  $\beta$  and let  $P_1$  and  $P_2$  be the corresponding points of impact. By considering the quantities  $(\beta_1^2 + \beta_2^2)$  and  $\beta_1^2\beta_2^2$ , or otherwise, show that the linear distance between  $P_1$  and  $P_2$  is

$$2R \sin\left(\frac{\omega}{g}\sqrt{V^2 - Rg}\right).$$

**Section C: Probability and Statistics**

- 12** It is known that there are three manufacturers A, B, C, who can produce micro chip MB666. The probability that a randomly selected MB666 is produced by A is  $2p$ , and the corresponding probabilities for B and C are  $p$  and  $1 - 3p$ , respectively, where  $0 \leq p \leq \frac{1}{3}$ . It is also known that 70% of MB666 micro chips from A are sound and that the corresponding percentages for B and C are 80% and 90%, respectively.

Find in terms of  $p$ , the conditional probability,  $P(A|S)$ , that if a randomly selected MB666 chip is found to be sound then it came from A, and also the conditional probability,  $P(C|S)$ , that if it is sound then it came from C.

A quality inspector took a random sample of one MB666 micro chip and found it to be sound. She then traced its place of manufacture to be A, and so estimated  $p$  by calculating the value of  $p$  that corresponds to the greatest value of  $P(A|S)$ . A second quality inspector also took a random sample of one MB666 chip and found it to be sound. Later he traced its place of manufacture to be C and so estimated  $p$  by applying the procedure of his colleague to  $P(C|S)$ .

Determine the values of the two estimates and comment briefly on the results obtained.

- 13** A stick is broken at a point, chosen at random, along its length. Find the probability that the ratio,  $R$ , of the length of the shorter piece to the length of the longer piece is less than  $r$ .

Find the probability density function for  $R$ , and calculate the mean and variance of  $R$ .

- 14** You play the following game. You throw a six-sided fair die repeatedly. You may choose to stop after any throw, except that you must stop if you throw a 1. Your score is the number obtained on your last throw. Determine the strategy that you should adopt in order to maximize your expected score, explaining your reasoning carefully.