Cambridge International A Level

Cambridge Assessment International Education
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/11**

Paper 1 **October/November 2018**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the October/November 2018 series for most Cambridge IGCSE™, Cambridge International A and AS Level components and some Cambridge O Level components.

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[Turn over

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Abbreviations

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only – often written by a 'fortuitous' answer
ISW	Ignore Subsequent Working
SOI	Seen or implied
SR	Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1(i)	*EITHER:* $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 36 - 10 - 3\times 9 = -1 \neq 0$	**M1 A1**	Calculates determinant
	OR: $\alpha + 2\beta + 3\gamma = 0$ $2\alpha + 9\beta + 3\gamma = 0 \Rightarrow 2\beta - \gamma = 0$ $\alpha + 4\gamma = 0$ $9\beta - 5\gamma = 0$ $\Rightarrow \beta = 0 \Rightarrow \alpha = \gamma = 0$	**(M1A1)**	Solves homogeneous system of equations. Eliminates one variable.
	Therefore $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent (and span $\mathbb{R}^3$) so form a basis for $\mathbb{R}^3$.	**A1**	States either that vectors are linearly independent or that vectors span $\mathbb{R}^3$.
		3	
1(ii)	$l\begin{pmatrix}1\\2\\1\end{pmatrix} + m\begin{pmatrix}2\\9\\0\end{pmatrix} + n\begin{pmatrix}3\\3\\4\end{pmatrix} = \begin{pmatrix}0\\-8\\3\end{pmatrix}$ $\Rightarrow l = m = -1$ and $n = 1$.	**M1**	Sets up system of equations.
	$\Rightarrow \mathbf{d} = \mathbf{c} - \mathbf{b} - \mathbf{a}$	**A1**	
		2	

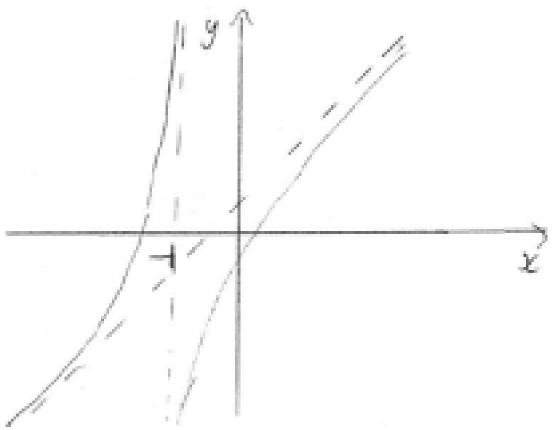
Question	Answer	Marks	Guidance
2(i)	$\alpha + 2\alpha + 4\alpha = -p$	**B1**	Sum of roots.
	$2\alpha^2 + 4\alpha^2 + 8\alpha^2 = q$	**B1**	Sum of products in pairs.
	$\frac{14\alpha^2}{7\alpha} = -\frac{q}{p}$	**M1**	Combines equations.
	$\Rightarrow 2p\alpha + q = 0$	**A1**	Verifies result (AG).
		4	
2(ii)	$8\alpha^3 = -r$	**B1**	Product of roots.
	$\Rightarrow r = \frac{q^3}{p^3} \Rightarrow p^3 r - q^3 = 0$	**B1**	Verifies result (AG).
		2	

Question	Answer	Marks	Guidance
3(i)	$u_1 < 3(\text{given})$	B1	States base case.
	Assume that $u_k < 3$	B1	States inductive hypothesis.
	Then $3 - u_{k+1} = 3 - \dfrac{4u_k + 9}{u_k + 4} = \dfrac{-u_k + 3}{u_k + 4} > 0 \Rightarrow u_{k+1} < 3$	M1 A1	
	Hence, by induction, $u_n < 3$ for all $n \geqslant 1$.	B1	States conclusion.
		5	
3(ii)	$u_{n+1} - u_n = \dfrac{4u_n + 9}{u_n + 4} - u_n = \dfrac{-u_n^2 + 9}{u_n + 4}$	M1 A1	Considers $u_{n+1} - u_n$.
	So $u_n < 3 \Rightarrow u_{n+1} - u_n > 0$.	B1	Uses $u_n < 3$
		3	

Question	Answer	Marks	Guidance
4(i)	$\frac{dx}{dt}=1-\cos 2t$ $\frac{dy}{dt}=2\sin t\cos t$	B1	B1 for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
	$\frac{dx}{dt}=1-\cos 2t=2\sin^2 t$	M1	Uses an appropriate identity.
	$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{4\sin^4 t+4\sin^2 t\cos^2 t}=2\sin t$	A1	
	$S=2\pi\int_0^{\pi}\left(\sin^2 t\right)\left(2\sin t\right)dt\Rightarrow a=4$	M1 A1	Uses the correct formula for S
		5	
4(ii)	$S=4\pi\int_0^{\pi}\sin^3 t\,dt=\pi\int_0^{\pi}3\sin t-\sin 3t\,dt$	M1	Uses the result $4\sin^3 t=3\sin t-\sin 3t$
	$=\pi\left[-3\cos t+\frac{1}{3}\cos 3t\right]_0^{\pi}$	A1	
	$=\pi\left(3-\frac{1}{3}-\left(-3+\frac{1}{3}\right)\right)=\frac{16\pi}{3}$	A1	Answer must be exact.
		3	

Question	Answer	Marks	Guidance
5(i)	$\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e} \Rightarrow \mathbf{Ae} + \mathbf{Be} = \lambda\mathbf{e} + \mu\mathbf{e}$	**M1**	Adds equations.
	$\Rightarrow (\mathbf{A} + \mathbf{B})\mathbf{e} = (\lambda + \mu)\mathbf{e}$	**A1**	
		2	
5(ii)	$\begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$	**M1**	Multiples matrix with vector.
	$\Rightarrow \lambda = 3$	**A1**	Finds one eigenvalue.
	$\lambda = 1, 2$	**A1**	Finds other two.
		3	
5(iii)	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{pmatrix}$	**B1**	
	$\mathbf{D} = \begin{pmatrix} 7^3 & 0 & 0 \\ 0 & 6^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 343 & 0 & 0 \\ 0 & 216 & 0 \\ 0 & 0 & 27 \end{pmatrix}$	**M1 A1**	Or correctly matched permutations of columns.
		3	

Question	Answer	Marks	Guidance
6(i)	Vertical asymptote is $x = -1$.	B1	
	$x^2 + ax - 1 = (x+1)(x+a-1) - a$ or $x+1\overline{)x^2+ax-1}$ with quotient $x+a-1$	M1	By inspection or long division.
	Thus the oblique asymptote is $y = x + a - 1$	A1	
		3	
6(ii)	$a^2 + 4 > 0$	B1	
		1	
6(iii)	$\frac{(x+1)(2x+a)-(x^2+ax-1)}{(x+1)^2} = 0 \Rightarrow x^2 + 2x + a + 1 = 0$ Discriminant $= 4 - 4(a+1) < 0$	M1	
	Therefore there are no stationary points on C.	A1	
		2	

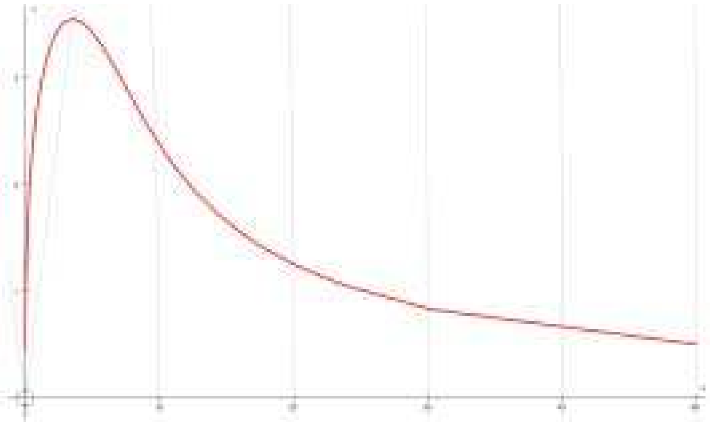
Question	Answer	Marks	Guidance
6(iv)		**B1**	Correct y-intercept and asymptotes drawn.
		B1 B1	Each branch correct
		3	

Question	Answer	Marks	Guidance
7(i)	Write $c=\cos\theta$, $s=\sin\theta$. $\cos 8\theta + i\sin 8\theta = (c+is)^8$	M1	Uses binomial theorem.
	$\Rightarrow \sin 8\theta = 8c^7 s - 56c^5 s^3 + 56c^3 s^5 - 8cs^7$	A1	
	$= 8cs\left(c^6 - 7c^4 s^2 + 7c^2 s^4 - s^6\right)$	M1	Factorises.
	$c^6 - 7c^4 s^2 + 7c^2 s^4 - s^6 = \left(1-s^2\right)^3 - 7\left(1-s^2\right)^2 s^2 + 7\left(1-s^2\right)s^4 - s^6$	M1	Uses $c^2 = 1 - s^2$.
	$= \left(1-3s^2+3s^4-s^6\right) - 7\left(s^2-2s^4+s^6\right) + 7\left(s^4-s^6\right) - s^6$ $= 1 - 10s^2 + 24s^4 - 16s^6$	A1	
	$\Rightarrow \sin 8\theta = 8cs\left(1-10s^2+24s^4-16s^6\right)$	A1	AG
		6	

Question	Answer	Marks	Guidance
7(ii)	$\sin 2\theta = 2cs \Rightarrow \dfrac{\sin 8\theta}{\sin 2\theta} = 4\left(1-10s^2+24s^4-16s^6\right)$	M1	Uses $\sin 2\theta = 2cs$ to relate with equation in part (i)
	$\sin 8\theta = 0 \Rightarrow \theta = \dfrac{n\pi}{8}$	M1	Solves $\sin 8\theta = 0$
	$x = \sin\dfrac{\pi}{8}$	A1	Gives one correct solution
	$\sin\dfrac{n\pi}{8}$, $n = \pm1, \pm2 \pm 3\, or\, 1,2,3,9,10,11$	A1	Gives five other solutions. Allow different values of k as long as six distinct solutions are found. $\sin 0$ and $\sin\dfrac{\pi}{2}$ must be excluded.
		4	

Question	Answer	Marks	Guidance
8(i)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$	M1	Finds normal to $\boldsymbol{\Pi}_1$.
	$-x + 8y - 4z = 3$	M1 A1	Uses point on plane to find cartesian equation, AEF.
		3	
8(ii)	$\begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 9\sqrt{11}\cos\theta$	M1	Uses scalar product.
	$\Rightarrow \theta = 72.5°$	A1	Accept 1.26 rad.
		2	
8(iii)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ -1 & 8 & -4 \end{vmatrix} = 4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$	M1 A1	Finds direction of line of intersection.
	Point on both planes is, e.g. $(1,1,1)$.	M1 A1	Finds point common to both planes.
	$\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$	A1	States vector equation of line.
		5	

Question	Answer	Marks	Guidance
9(i)	$\frac{25}{2}\int_{0.01}^{\frac{\pi}{2}} \cot\theta \, d\theta$	M1	Uses $\frac{1}{2}\int r^2 \, d\theta$.
	$= \frac{25}{2}[\ln\sin\theta]_{0.01}^{\frac{\pi}{2}}$	A1	
	$= -\frac{25}{2}\ln\sin 0.01 \approx 57.6$	A1	
		3	
9(ii)	$y = 5\cos^{\frac{1}{2}}\theta\sin^{\frac{1}{2}}\theta = \frac{5}{\sqrt{2}}\sin^{\frac{1}{2}}2\theta$	M1	Uses $y = r\sin\theta$.
	$\theta = 0.01 \Rightarrow y \approx 0.5$	A1	
		2	
9(iii)	$\frac{dy}{d\theta} = \frac{5}{\sqrt{2}}\sin^{-\frac{1}{2}}2\theta\cos 2\theta = 0$ or $\max(\sin 2\theta) = 1$	M1 A1	Sets $\frac{dy}{d\theta} = 0$ or considers max (AEF).
	$\Rightarrow y = \frac{5\sqrt{2}}{2}$ $(= 3.54)$	A1	
		3	

Question	Answer	Marks	Guidance
9(iv)		**B1**	Intersecting the initial line only when $x = 0$ and $y = 0$.
		B1	Correct shape.
		2	

Question	Answer	Marks	Guidance
10(i)	$m^2+2m+10=0 \;\Rightarrow\; m=-1\pm 3i.$	M1	Finds complementary function.
	$x=e^{-t}\left(A\cos 3t+B\sin 3t\right)$	A1	
	$x=p\cos 3t+q\sin 3t$	M1	Finds particular integral.
	$\Rightarrow \dot{x}=-3p\sin 3t+3q\cos 3t \Rightarrow \ddot{x}=-9p\cos 3t-9q\sin 3t$	A1	
	$\left(p+6q\right)\cos 3t+\left(q-6p\right)\sin 3t=37\sin 3t$	M1	
	$\Rightarrow\; p=-6,\quad q=1$ $x=e^{-t}\left(A\cos 3t+B\sin 3t\right)-6\cos 3t+\sin 3t$	A1	
	$x=3$ when $t=0 \;\Rightarrow A-6=3\Rightarrow A=9$	B1	Uses initial conditions.
	$\dot{x}=e^{-t}\left(-3A\sin 3t+3B\cos 3t\right)-e^{-t}\left(A\cos 3t+B\sin 3t\right)+18\sin 3t+3\cos 3t$	M1 A1	
	$\dot{x}=0$ when $t=0\Rightarrow 3B-A+3=0\Rightarrow B=2$ $x=e^{-t}\left(9\cos 3t+2\sin 3t\right)-6\cos 3t+\sin 3t$	A1	Obtains solution, AEF.
		10	
10(ii)	As $t\to\infty$, $e^{-t}\to 0 \quad\Rightarrow x\approx -6\cos 3t+\sin 3t$	B1	Obtains limit.
	$\sin 3t-6\cos 3t=\sqrt{37}\left(\frac{1}{\sqrt{37}}\sin 3t-\frac{6}{\sqrt{37}}\cos 3t\right)$	M1	Converts to $R\sin\left(3t-\phi\right)$
	$=\sqrt{37}\sin\left(3t-\tan^{-1}6\right).$	A1	AG
		3	

Question	Answer	Marks	Guidance
11E(i)	$(2r+1)^2-(2r-1)^2=8r$	B1	
	$\Rightarrow\ 8\sum_{r=1}^{n} r=-1^2+(2n+1)^2$	M1	Sums both sides and uses method of differences.
	$\Rightarrow \sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$	A1	AG.
		3	
11E(ii)	$(2r+1)^4-(2r-1)^4=\left((2r+1)^2+(2r-1)^2\right)\left((2r+1)^2-(2r-1)^2\right)$	M1	Uses difference of squares or expands.
	$=(8r^2+2)(8r)=16(4r^3+r)$	A1	
	$\Rightarrow 4\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=-\left(1-\frac{1}{2}\right)^4+\left(n+\frac{1}{2}\right)^4$	M1	Sums both sides and uses method of differences.
	$4\sum_{r=1}^{n} r^3=\left(n+\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^4-\frac{1}{2}n(n+1)$	M1	Uses formula for $\sum r$
	$=\left(\left(n+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right)\left(\left(n+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)-\frac{1}{2}n(n+1)=n^2(n+1)^2$	A1	AG
		5	
11E(iii)	$S=\frac{1}{4}(2N+1)^2(2N+2)^2=(2N+1)^2(N+1)^2$	B1	Uses formula for $\sum r^3$. AG.
		1	

Question	Answer	Marks	Guidance
11E(iv)	$T = S - \sum_{r=1}^{N}(2r)^3 = (2N+1)^2(N+1)^2 - 2N^2(N+1)^2$	M1	Eliminates even terms from S.
	$(N+1)^2(2N^2+4N+1)$	A1	Accept $(N+1)^2(2N(N+2)+1)$.
		2	
11E(v)	$\frac{S}{T} = \frac{(2N+1)^2}{2N^2+4N+1} = \frac{4N^2+4N+1}{2N^2+4N+1}$	M1	Writes fraction as quadratic in N divided by quadratic in N.
	Converges to 2 as $N \to \infty$.	A1	
		2	
11O(i)	$x = -1 \Rightarrow y^3 + 2y - 3 = 0 \Rightarrow (y-1)(y^2+y+3) = 0$	M1	Considers cubic polynomial in y.
	There is only one real root ($=1$).	A1	
		2	

Question	Answer	Marks	Guidance
11O(ii)	$2x+2\left(x\frac{dy}{dx}+y\right)$	**B1**	Differentiates LHS correctly.
	$=3y^2\frac{dy}{dx}$	**B1**	Differentiates RHS correctly.
	$\Rightarrow\left(3y^2-2x\right)\frac{dy}{dx}=2x+2y$ $x=-1, y=1\Rightarrow\frac{dy}{dx}=0.$	**B1**	Substitutes $(-1,1)$, AG.
		3	
11O(iii)	$\left(3y^2-2x\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(6y\frac{dy}{dx}-2\right)=2+2\frac{dy}{dx}$	**M1 M1**	Differentiates equation again.
	$x=-1, y=1, \frac{dy}{dx}=0\Rightarrow\frac{d^2y}{dx^2}=\frac{2}{5}$	**A1**	Substitutes $(-1,1)$ and $\frac{dy}{dx}=0$, AG.
		3	
11O(iv)	$\int_{-1}^{0}x^n\frac{d^ny}{dx^n}dx=\left[x^n\frac{d^{n-1}y}{dx^{n-1}}\right]_{-1}^{0}-n\int_{-1}^{0}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}dx$	**M1 A1**	Integrates by parts.
	$=(-1)^{n+1}A_{n-1}-nI_{n-1}.$	**A1**	AG.
		3	

Question	Answer	Marks	Guidance
11O(v)	$I_3 = (-1)^4 A_2 - 3I_2 = A_2 - 3(-A_1 - 2I_1) = \frac{2}{5} + 6I_1$	M1 A1	Uses reduction formulae.
		2	